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TWO VARIABLE LOGIC WITH ULTIMATELY PERIODIC COUNTING*

MICHAEL BENEDIKT†, EGOR V. KOSTYLEV‡, AND TONY TAN§

Abstract. We consider the extension of FO^2 with quantifiers that state that the number of elements where a formula holds should belong to a given ultimately periodic set. We show that both satisfiability and finite satisfiability of the logic are decidable. We also show that the spectrum of any sentence, i.e., the set of the sizes of its finite models, is definable in Presburger arithmetic. In the process we present several refinements to the "biregular graph method." In this method, decidability issues concerning two-variable logics are reduced to questions about Presburger definability of integer vectors associated with partitioned graphs, where nodes in a partition satisfy certain constraints on their in- and out-degrees.

Key words. Presburger arithmetic, two-variable logic

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1. Introduction.

In the search for expressive logics with decidable satisfiability problem, two-variable logic, denoted here as FO^2, is one yardstick. This logic is expressive enough to subsume basic modal logic and many description logics, while satisfiability and finite satisfiability for this logic coincide, and both are decidable [31, 22, 14]. However, FO^2 lacks the ability to count. Two-variable logic with counting, C^2, is a decidable extension of FO^2 that adds *counting quantifiers.* In C^2 one can write, for example, formulas $\exists^5 x\, P(x)$ and $\forall x \exists^{\geqslant 5} y\, E(x,y)$ which, respectively, express that there are exactly 5 elements in unary relation P, and that every element in a graph has at least 5 adjacent edges. Satisfiability and finite satisfiability do not coincide for C^2, but both are decidable [15, 23]. In [23] the problems were shown to be NEXPTIME-complete under a unary encoding of numbers, and this was extended to binary encoding in [25]. However, the numerical capabilities of C^2 are quite limited. For example, one cannot express that the number of outgoing edges of each element in the graph is even.

A natural extension is to combine FO^2 with *Presburger arithmetic* where one is allowed to define collections of tuples of integers from addition and equality using Boolean operators and quantifiers. The collections of k-tuples that one can define in this way are the *semilinear sets,* and the collections of integers (when $k = 1$) definable

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†University of Oxford, Oxford, England (michael.benedikt@cs.ox.ac.uk).
‡University of Oslo, Oslo, Norway (egor.kostylev@cs.ox.ac.uk).
§University of Liverpool, Liverpool, England (tonytan@liverpool.ac.uk).

are the *ultimately periodic sets.* It is natural to consider the addition of *Presburger quantification* to fragments of two-variable logic; this is in the spirit of works such as [4, 2]. For every definable set $\phi(x,y)$ and every ultimately periodic set S, one has a formula $\exists^S y\, \phi(x,y)$ that holds at x when the number of y such that $\phi(x,y)$ is in S. We let $\mathsf{FO}^2_{\mathrm{Pres}}$ denote the logic that adds this construct to FO^2.

On the one hand, the corresponding quantification over general k-tuples (allowing semilinear rather than only ultimately periodic sets) easily leads to undecidability [16, 3]. On the other hand, adding this quantification to modal logic has been shown to preserve decidability [1, 10]. Related *one-variable fragments* in which we have only a unary relational vocabulary and the main quantification is $\exists^S x\, \phi(x)$ are known to be decidable (see, e.g., [2]), and their decidability is the basis for a number of software tools focusing on integration of relational languages with Presburger arithmetic [21]. The decidability of full $\mathsf{FO}^2_{\mathrm{Pres}}$ is, to the best of our knowledge, open. There are a number of other extensions of C^2 that have been shown decidable; for example, it has been shown that one can allow a distinguished equivalence relation [29] or a forest-structured relation [9, 7]. $\mathsf{FO}^2_{\mathrm{Pres}}$ is easily seen to be orthogonal to these other extensions. For example, equivalence relations and forest-structure are not expressible in $\mathsf{FO}^2_{\mathrm{Pres}}$, whereas modulo counting is not expressible in the logics of [29, 9, 7].

In this paper we show that both satisfiability and finite satisfiability of $\mathsf{FO}^2_{\mathrm{Pres}}$ are decidable. Our result uses a method based on analyzing *biregular graph constraints,* introduced for analyzing C^2 in [19]. In this analysis we search for the existence of graphs equipped with a partition of vertices based on constraints on the out- and in-degree. Such a partitioned graph can be characterized by the cardinalities of each partition component, and the key step in showing these decidability results is to prove that the set of tuples of integers representing valid sizes of partition components is definable by a formula in Presburger arithmetic. From this "biregular graph constraint Presburger definability" result, one can reduce satisfiability in the logic to satisfiability of a Presburger formula, and from there infer decidability using known results on Presburger arithmetic. We will also use this method to get information on the *spectrum* of a $\mathsf{FO}^2_{\mathrm{Pres}}$ sentence: the set of sizes of models of the sentence. We use the method to conclude that this set is definable in Presburger arithmetic, a result that had been demonstrated for C^2 in [19].

Organization. Section 2 provides background on two-variable logic and Presburger arithmetic. Section 3 introduces our major results on the logic, and gives a reduction of these logic-based problems to results concerning the analysis of constrained biregular graphs. Section 4 gives some of the details behind the core lemmas concerning Presburger definability of solutions to biregular graph problems that underlie the proof, and provides a full proof in the case where there is only a single kind of edge in the graph. We refer to this as the "1-color case." Section 5 generalizes to give a proof in the case of an unbounded number of edge colors, but with an extra restriction on the matrices that specify the graph constraints. The restriction is that they are "simple matrices." Section 6 extends the analysis in section 5 to the complete graph cases—but still with the restriction on simple matrices. Section 7 shows how to reduce the general case to the simple case. Section 8 provides complexity upper bounds for all problems considered in this paper. Section 9 gives an application of the graph analysis result to the spectrum problem. After a discussion of related work in section 10, the paper closes with conclusions and future directions in section 11. Some proofs that are not required in order to follow the main line of argument in the paper are deferred to the appendix. In addition, to make the main line of argument clearer, we consider only the finite graph case in the body of the paper, which already implies decidability of the finite satisfiability of $\mathsf{FO}^2_{\mathrm{Pres}}$. The general case is deferred to the appendix.

2. Preliminaries. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and let $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$.

Linear and ultimately periodic sets. A set of the form $\{a + ip \mid i \in \mathbb{N}\}$, for some $a, p \in \mathbb{N}$, is a *linear set.* We will denote such a set by a^{+p}, where a and p are called the *offset* and *period* of the set, respectively. Note that, by definition, $a^{+0} = \{a\}$, which is a linear set. For convenience, we define $\emptyset$ and $\{\infty\}$ (which may be written as ∞^{+p}) to also be linear sets. An ultimately periodic set (*u.p.s.*) S is a finite union of linear sets.

In this paper we represent a u.p.s. $S = \{c_1\} \cup \cdots \cup \{c_m\} \cup a_1^{+p_1} \cup \cdots \cup a_n^{+p_n}$, where $p_1, \ldots, p_n \neq 0$, as a "finite set" $\{c_1, \ldots, c_m, a_1^{+p_1}, \ldots, a_n^{+p_n}\}$. In such a representation the offsets in S are $c_1, \ldots, c_m, a_1, \ldots, a_n$ and the (nonzero) periods are $p_1, \ldots, p_n$. For an integer a, we write $a \in S$, if a is in S in the standard sense. Abusing notation, we write $a^{+p} \in S$, if $a + ip \in S$ for every $i \in \mathbb{N}$. We also note that the set of u.p.s.'s is closed under complement, union, and intersection [13].

Two-variable logic with ultimately periodic counting quantifiers. An *atomic formula* is one of the following:

- an atom $R(\vec{u})$, where R is a predicate, and $\vec{u}$ is a tuple of variables/constants of appropriate size;
- an equality $u = u'$ with u and u' variables/constants;
- one of the formulas $\top$ and $\bot$ denoting the True and False values.

The logic $\mathsf{FO}^2_{\mathrm{Pres}}$ is the class of formulas using only variables x and y, built up from atomic formulas and equalities using the usual Boolean connectives and also *ultimately periodic counting quantification,* which is of the form $\exists^S x\ \phi$, where S is a u.p.s. and ϕ is an $\mathsf{FO}^2_{\mathrm{Pres}}$ formula. One special case is where S is a singleton $\{a\}$ with $a \in \mathbb{N}_\infty$, which we write $\exists^a x\ \phi$; in the case of $a \in \mathbb{N}$, these are counting quantifiers. The semantics of $\mathsf{FO}^2_{\mathrm{Pres}}$ is defined as usual except that, for every $a \in \mathbb{N}$, $\exists^a x\ \phi$ holds when there are *exactly* a number of x's such that ϕ holds, $\exists^\infty x\ \phi$ holds when there are infinitely many x's such that ϕ holds, and $\exists^S x\ \phi$ holds when there is some $a \in S$ such that $\exists^a x\ \phi$ holds.

Formulas in $\mathsf{FO}^2_{\mathrm{Pres}}$ still use only two variables. So just as in FO^2 they can be normalized. If they use atomic predicates with arity 3 or above, they can be rewritten into an equisatisfiable formula that uses only unary and binary predicates. See [14, sect. 3] or [30] for the details of such a rewriting. In addition, each constant c can be represented with a fresh unary predicate U_c that contains exactly one element. For constants c_1, c_2, an atomic predicate $x = c_1$ can then be rewritten as $U_{c_1}(x)$, and predicate $c_1 = c_2$ can be rewritten as $\forall x\ U_{c_1}(x) \leftrightarrow U_{c_2}(x)$. Thus, in this paper we may assume that $\mathsf{FO}^2_{\mathrm{Pres}}$ formulas use only unary and binary predicates, and do not use constants.

Note that when S is $0^{+1} \cup \{\infty\} = \mathbb{N}_\infty$, $\exists^S x\ \phi$ is equivalent to $\top$. When S is $0^{+1} = \mathbb{N}$, $\exists^S x\ \phi$ semantically means that there are finitely many x such that ϕ holds. We also observe that, for every formula ϕ, $\exists^\emptyset x\ \phi$ is equivalent to $\bot$, $\exists^0 x\ \phi$ is equivalent to $\forall x\ \neg\phi$, and $\neg\exists^S x\ \phi$ is equivalent to $\exists^{\mathbb{N}_\infty - S} x\ \phi$. We remark that $\mathbb{N}_\infty - S$ is a u.p.s., whenever S is a u.p.s.

For example, we can state in $\mathsf{FO}^2_{\mathrm{Pres}}$ that a graph is undirected and every node in a graph has even degree (i.e., the graph is Eulerian in the sense that every connected component has Eulerian cycle):

$$\forall x \forall y\ E(x,y) \leftrightarrow E(y,x) \quad \wedge \quad \forall x \exists^S y\ E(x,y) \wedge x \neq y, \qquad \text{where } S = 2^{+2}.$$

Clearly $\mathsf{FO}^2_{\mathrm{Pres}}$ extends C^2, the fragment of the logic where only counting quantifiers are used, and FO^2, the fragment where only the classical quantifier $\exists x$ is allowed (which is equivalent to $\exists^S x$ for $S = \{1^{+1}, \infty\}$).

Presburger arithmetic. An *existential Presburger formula* is a first-order logic formula of the form $\exists x_1 \cdots \exists x_k\ \phi$, where ϕ is a quantifier-free formula over the signature including constants $0, 1$, a binary function symbol $+$, and a binary relation $\leqslant$. Such a formula is a *sentence* if it has no free variables. The notion of a sentence holding in a structure interpreting the function, relation, and constants is defined in the usual way. The structure $\mathcal{N} = (\mathbb{N}, +, \leqslant, 0, 1)$, is defined by interpreting $+, \leqslant, 0, 1$ in the standard way. We will focus not on this structure, but on $\mathcal{N}_\infty = (\mathbb{N}_\infty, +, \leqslant, 0, 1)$ which is the same as $\mathcal{N}$, except that there is an element ∞, with $a + \infty = \infty$ and $a \leqslant \infty$ for each $a \in \mathbb{N}_\infty$. Note that in $\mathcal{N}_\infty$ there is a unique element n such that $n + 1 = n$, namely, ∞. We will thus abuse notation in the followingby writing $t = \infty$, where t is a term, as syntactic sugar for $t = t + 1$. Since $\mathcal{N}$ is quantifier-free definable in $\mathcal{N}_\infty$, satisfaction of a formula in finite integers can still be expressed when working over $\mathcal{N}_\infty$.

It is known that the problem of checking whether an existential Presburger sentence holds in $\mathcal{N}$ is decidable and is NP-complete [24]. Further, the analogous problem for $\mathcal{N}_\infty$ can easily be reduced to that for $\mathcal{N}$. Indeed, we can first guess which variables are mapped to ∞ and then which atoms should be true. Then we can check whether each guessed atomic truth value is consistent with other guesses, in the sense that no two contradicting atoms are guessed to be both true or false at the same time. We can determine additional variables which must be infinite based on this choice. Finally we can restrict ourselves to atoms that do not involve variables guessed to be infinite, and check that the conjunction is satisfiable for $\mathcal{N}$. This gives us the following theorem.

THEOREM 2.1. *The problems of checking whether an existential Presburger sentence holds in $\mathcal{N}_\infty$ in NP.*

3. From Analysis of Constrained Regular Graph Problems to Decidability of $\mathsf{FO}^2_{\mathrm{Pres}}$.

In this section we prove decidability of $\mathsf{FO}^2_{\mathrm{Pres}}$ satisfiability, with the decidability following from a result on Presburger definability of collection of integers that characterizes graphs satisfying a set of degree constraints. These definability results will be proven later in the paper. Our decision procedure is based on the key notion of biregular graphs. Note that whenever we talk about graphs or digraphs (i.e., directed graphs), by default we allow both finite and infinite sets of vertices and edges.

3.1. Biregular graphs and constrained biregular graph problems.

We fix an integer $p \geqslant 0$. Let $\mathbb{N}_{\infty,+p}$ denote the set $\mathbb{N}_\infty \cup \{a^{+p} \mid a \in \mathbb{N}_\infty\}$. For integers $t, m \geqslant 1$, let $\mathbb{N}^{t \times m}_{\infty,+p}$ denote the set of matrices with t rows and m columns where each entry is an element of $\mathbb{N}_{\infty,+p}$. For an integer $k \geqslant 1$, let $[k]$ denote the set $\{1, 2, \ldots, k\}$.

A *t-color bipartite (undirected) graph* is $G = (U, V, E_1, \ldots, E_t)$, where U and V are sets of vertices, and $E_1, \ldots, E_t$ are pairwise disjoint sets of edges between U and V—that is, pairs $(u, v) \in U \times V$. Edges in E_i are called *E_i-edges,* and we often refer to an index from 1 to t—the type of an edge—as a *color.* For a vertex $u \in U \cup V$, the *E_i-degree* of u is the number of E_i-edges adjacent to u. The degree of u is the sum of the E_i-degrees for $i = 1, \ldots, t$: we use this primarily for brevity when the there is only a single edge relation. In the context of multiple relations, we sometimes refer to this as the *total degree* to emphasize that all relations are considered. We say that G is *complete* if $U \times V = \bigcup_{i=1}^{t} E_i$.

For two matrices $A \in \mathbb{N}_{\infty,+p}^{t\times m}$ and $B \in \mathbb{N}_{\infty,+p}^{t\times n}$, graph G is an $A|B$-*biregular* graph, if there exist a partition[1] $U = U_1 \uplus \cdots \uplus U_m$ and a partition $V = V_1 \uplus \cdots \uplus V_n$ such that for every $i \in [t]$, for every $k \in [m]$, and for every $\ell \in [n]$, the E_i-degree of every vertex in U_k is $A_{i,k}$ (i.e., the element of A in the ith row and kth column) and the E_i-degree of every vertex in V_ℓ is $B_{i,\ell}$; note here that, by abuse of notation, when we say that a nonnegative integer z is a linear set a^{+p}, we mean that $z \in a^{+p}$. For each such partition, we say that G *has size* $\bar{M}|\bar{N}$, where $\bar{M} = (|U_1|, \ldots, |U_m|)$ and $\bar{N} = (|V_1|, \ldots, |V_n|)$. The partitions $U = U_1 \uplus \cdots \uplus U_m$ and $V = V_1 \uplus \cdots \uplus V_n$ are called a *witness partition* for $A|B$-biregularity. We should remark that some U_i and V_i are allowed to be empty. The matrices A and B are called (t-*color*) *degree matrices* and the vectors $\bar{M}$ and $\bar{N}$ are called *size vectors.* For convenience, we treat the empty graph (i.e., the graph with no vertex) as a complete $A|B$-biregular graph for any degree matrices A and B.

The above definitions can be easily adapted for the case of directed graphs that are not necessarily bipartite. A t-color *directed graph* (or *digraph*) is a tuple $G = (V, E_1, \ldots, E_t)$, where $E_1, \ldots, E_t$ are pairwise disjoint sets of directed edges on a set V of vertices such that (i) there are no self-loops—that is, $(v, v) \notin E_i$ for every $v \in V$ and every E_i, and (ii) if $(u, v) \in E_i$, then $(v, u) \notin E_j$ for every E_j. As before, edges in E_i are called E_i-edges. The E_i-indegree and -outdegree of a vertex u is defined as the number of incoming and outgoing E_i-edges incident to u. We say that G is *complete,*if, for every $u, v \in V$ and $u \neq v$, either (u, v) or (v, u) is an E_i-edge, for some E_i. We consider the empty digraph and the digraph with only one vertex without any edge as complete digraphs.

We say that G is an $A|B$-*regular* digraph, for $A, B \in \mathbb{N}_{\infty,+p}^{t\times m}$, if there exists a partition $V = V_1 \uplus \cdots \uplus V_n$ such that, for every $i \in [t]$ and for every $k \in [m]$, the E_i-outdegree and -indegree of every vertex in V_k is $A_{i,k}$ and $B_{i,k}$, respectively. We say that G has size $(|V_1|, \ldots, |V_m|)$, and call $V = V_1 \uplus \cdots \uplus V_m$ a witness partition for $A|B$-regularity of G. When the entries in A and B are all 0 or 0^{+p}, we regard the graph with only one vertex to be a complete $A|B$-regular digraph.

In this work we will be interested in computational problems concerning the possible sizes of an $A|B$-biregular graph or -regular digraph, and the possible sizes of a complete $A|B$-biregular graph or -regular digraph. Biregular one-color graphs are arguably quite natural, independently of any connection with satisfiability of a logic. Completeness, as well as disjointness of edges for different colors, is more motivated specifically by our application to logic. Intuitively, the different edge colors in a biregular graph represent the possible relationships between two elements in a structure. One color might represent a binary relationship, and another might represent its negation. Since every two elements have *some* relationship, we want all pairs to be colored by exactly one edge color. This will be formalized in subsection 3.2 below.

We briefly consider the (finite) *membership problem*: given size vectors $\bar{M}, \bar{N}$ along with matrices A and B, all without ∞, decide if there is an $A|B$-biregular graph G with size $\bar{M}|\bar{N}$. The problem is clearly in NP if the entries in $\bar{M}$ and $\bar{N}$ are in unary, since we can guess G and check that it is $A|B$-biregular with size $\bar{M}|\bar{N}$.

The *degree sequence* for a (1-color) bipartite graph (U, V, E) with k vertices in U and k' vertices in V, is the pair of sequences $d_1, \ldots, d_k$ and $d'_1, \ldots, d'_{k'}$ where $d_1, \ldots, d_k$

[1] As usual, we write $U = U_1 \uplus \cdots \uplus U_m$ to denote the partition of U into the sets $U_1, \ldots, U_m$, i.e., when $U = U_1 \cup \cdots \cup U_m$ for pairwise disjoint sets $U_1, \ldots, U_m$.

enumerates the degrees of elements in U in nondecreasing order and $d'_1,\dots,d'_{k'}$ enumerates the degrees of elements of V in nondecreasing order.

It follows from the Gale–Reyser theorem (the main theorem in [20]) that one can determine in polynomial time whether a pair of sequences is the degree sequence of a bipartite graph. From this we derive the following.

PROPOSITION 3.1. *In the case of* 1*-color degree matrices with only entries from* $\mathbb{N}$*, coded in unary, the membership problem is in PTIME.*

Proof. The algorithm will first generate a pair of sequences that will be a degree sequence of any $A|B$-biregular graph with sizes $\bar{M}|\bar{N}$. We can do this in linear time: if an entry with fixed degree d is to have size m, the degree sequences will contain a contiguous subsequence consisting of m d's. We then apply Gale–Reyser to this sequence. ∎

While we will not provide a detailed analysis of the complexity of the membership problem, we will show that, when fixing A and B, we can succinctly describe—and hence efficiently compute—the size vectors of partitioned graphs for which membership holds. This will be a consequence of the following theorem.

THEOREM 3.2. *For all degree matrices* $A \in \mathbb{N}^{t\times m}_{\infty,+p}$ *and* $B \in \mathbb{N}^{t\times n}_{\infty,+p}$*, there is an (effectively computable) existential Presburger formula* $\mathsf{c\text{-}bireg}_{A|B}(\bar{x},\bar{y})$ *such that, for every pair of size vectors* $\bar{M} \in \mathbb{N}^m_\infty$ *and* $\bar{N} \in \mathbb{N}^n_\infty$*, the formula* $\mathsf{c\text{-}bireg}_{A|B}(\bar{M},\bar{N})$ *holds in* $\mathcal{N}_\infty$ *if and only if there is a complete* $A|B$*-biregular graph with size* $\bar{M}|\bar{N}$*.*

We have an analogous theorem for digraphs.

THEOREM 3.3. *For every pair of degree matrices* $A \in \mathbb{N}^{t\times m}_{\infty,+p}$ *and* $B \in \mathbb{N}^{t\times m}_{\infty,+p}$*, there is an (effectively computable) existential Presburger formula* $\mathsf{c\text{-}reg}_{A|B}(\bar{x})$ *such that for every size vector* $\bar{M} \in \mathbb{N}^m_\infty$*, the formula* $\mathsf{c\text{-}reg}_{A|B}(\bar{M})$ *holds in* $\mathcal{N}_\infty$ *if and only if there is a complete* $A|B$*-regular digraph with size* $\bar{M}$*.*

The proofs of these two theorems are given later in sections 4–7, beginning with an overview of the ideas via an extremely special case (the 1-color case) in section 4. An immediate consequence of these results is the decidability of graph analysis problems.

COROLLARY 3.4. *We can decide, given matrices* $A \in \mathbb{N}^{t\times m}_{\infty,+p}$ *and* $B \in \mathbb{N}^{t\times n}_{\infty,+p}$*, whether there exists a complete* $A|B$*-biregular graph. The analogous result holds for digraphs. Moreover, the decision procedure runs in nondeterministic exponential time in the size of* A *and* B *where the coefficients are written in binary.*

Proof. By Theorems 3.2 and 3.3, we can reduce the graph existence problems to checking whether the existential closures of $\mathsf{c\text{-}bireg}_{A|B}(\bar{x},\bar{y})$ and $\mathsf{c\text{-}reg}_{A|B}(\bar{x})$ hold in $\mathcal{N}_\infty$. In turn, these problems are decidable by Theorem 2.1. Moreover, the upper bound for both cases holds by Lemma 8.1, which we prove in section 8. ∎

Remark 3.5. Theorems 3.2 and 3.3, as well as Corollary 3.4, can be easily adjusted in the case where we are interested only in finite sizes, i.e., when $\bar{M} \in \mathbb{N}^m$ and $\bar{N} \in \mathbb{N}^n$, by replacing every atom $x = \infty$ in the formulas with the False value $\perp$ and requiring them to hold in $\mathcal{N}$, instead of $\mathcal{N}_\infty$. Alternatively, we can also state inside the formulas that none of the variables in $\bar{x}$ and $\bar{y}$ are equal to ∞.

The rest of this section will be devoted to proving the decidability result concerning our logic, making use of these theorems.

3.2. Reducing satisfiability in the logic to biregular graph problems. We are now ready to present the decidability result for two-variable logic with ultimately periodic quantifiers.

THEOREM 3.6. *For every* $\mathsf{FO}^2_{\mathrm{Pres}}$ *sentence* ϕ*,* (i) *there is an (effectively computable) existential Presburger sentence* $\mathsf{PRES}^{\infty}_{\phi}$ *such that* ϕ *has a model if and only if* $\mathsf{PRES}^{\infty}_{\phi}$ *holds in* $\mathcal{N}_\infty$ *and* (ii) *there is an (effectively computable) existential Presburger sentence* PRES_ϕ *such that* ϕ *has a finite model if and only if* PRES_ϕ *holds in* $\mathcal{N}$*.*

From the decision procedure for existential Presburger formulas (Theorem 2.1) mentioned in section 2, we will immediately obtain the following corollary.

COROLLARY 3.7. *Both satisfiability and finite satisfiability for* $\mathsf{FO}^2_{\mathrm{Pres}}$ *are decidable.*

We prove Theorem 3.6 using Theorems 3.2 and 3.3. We start by observing that satisfiability for an $\mathsf{FO}^2_{\mathrm{Pres}}$ sentence—as well as spectrum analysis, to be defined formally in section 9—can be converted effectively into the same question for a sentence in a variant of Scott normal form:

$$\phi \;:=\; \forall x \forall y\; \alpha(x,y) \;\wedge\; \bigwedge_{i=1}^{k} \forall x \exists^{S_i} y\; \beta_i(x,y) \wedge x \neq y, \tag{3.1}$$

where $\alpha(x,y)$ is a quantifier-free formula, each $\beta_i(x,y)$ is an atomic formula, and each S_i is a u.p.s. More precisely, every $\mathsf{FO}^2_{\mathrm{Pres}}$ sentence can be converted effectively into a sentence in the form (3.1) such that they are equisatisfiable and have the same spectrum. The proof, which is fairly standard, can be found in the appendix. By taking the least common multiple, we may assume that all the nonzero periods in all S_i are the same. For example, if $S_1 = \{0^{+2}\}$ and $S_2 = \{0^{+3}\}$, they can be rewritten as $S_1 = \{0^{+6}, 2^{+6}, 4^{+6}\}$ and $S_2 = \{0^{+6}, 3^{+6}\}$. Here it is worth mentioning that when we write $\alpha(x,y)$ and $\beta(x,y)$, we implicitly assume that both x and y occur. For the rest of this section, we fix an $\mathsf{FO}^2_{\mathrm{Pres}}$ sentence ϕ in the form (3.1), with all S_i as described above. The signature of structures we consider will be the signature of ϕ.

We recall some standard terminology. A 1*-type* is a maximally consistent set of atomic and negated atomic formulas using only variable x, including atomic formulas such as $r(x,x)$ or $\neg r(x,x)$. Each 1-type can be identified with the quantifier-free formula formed as the conjunction of its constituent formulas. Thus, we say that an element u in a structure $\mathcal{A}$ has 1-type π, if π holds on the element u. For a structure $\mathcal{A}$ with domain A, we let A_π denote the set of elements in $\mathcal{A}$ with 1-type π. Clearly A is partitioned into the sets A_π with π ranging over 1-types. Similarly, a 2*-type* is a maximally consistent set of binary atoms and negations of atoms containing $x \neq y$, where each atom or its negation uses two variables x and y.[2] The notion of a pair of elements (u,v) in a structure $\mathcal{A}$ having 2-type μ is defined as for 1-types. We let $\Pi = \{\pi_1, \pi_2, \ldots, \pi_n\}$ and $\mathcal{E} = \{\mu_1, \ldots, \mu_t\}$ denote the sets of all 1-types and 2-types (over the same signature as ϕ), respectively.

We can now explain the connection between satisfiability in the logic and graph analysis. This will involve associating with a model $\mathcal{A}$ for a formula ϕ a collection of graphs and digraphs, along with partitions that witnesses biregularity of the graphs and digraphs. The following crucial definition explains the first aspect, how to go from a structure $\mathcal{A}$ to a collection of graphs and digraphs.

[2]Under standard definitions, such as the ones in [14, 25], a 2-type may contain unary atoms or negations of unary atoms involving variable x or y. In this paper we use a different definition and require that each atom and the negation of an atom in a 2-type explicitly mentions both x and y.

DEFINITION 3.8. *Let $\mathcal{A}$ be a structure. A* graph representation *of $\mathcal{A}$ is a complete t-color digraph $G_{\mathcal{A}} = (V, E_1, \ldots, E_t)$, where the vertices in $G_{\mathcal{A}}$ are the elements in the domain of $\mathcal{A}$ and for each pair of elements (u, v), where $u \neq v$, we put an arbitrary orientation between them: either from u to v or from v to u. For each $i \in [t]$, the set of edges E_i is the set of pairs (u, v) where the orientation is from u to v and the 2-type of (u, v) is μ_i. We often denote the graph representation $G_{\mathcal{A}}$ as $G_{\mathcal{A}} = (V, \mu_1, \ldots, \mu_t)$, and we call a pair (u, v) a μ_i-edge, if its 2-type is μ_i.*

For a graph representation $G_{\mathcal{A}}$ of a structure $\mathcal{A}$, we will consider two kinds of subgraphs of $G_{\mathcal{A}}$. The first is the subgraph of $G_{\mathcal{A}}$ induced by the set A_π for a 1-type π, denoted by $G_{\mathcal{A},\pi}$. The second is the bipartite restriction of $G_{\mathcal{A}}$ on the vertices in A_π and $A_{\pi'}$, for different 1-types π, π', denoted by $G_{\mathcal{A},\pi,\pi'}$. That is, $G_{\mathcal{A},\pi,\pi'}$ is the complete (directed) bipartite graph, where A_π is the set of vertices on the left-hand side, $A_{\pi'}$ is the set of vertices on the right-hand side and the edges are between the vertices in A_π and the vertices in $A_{\pi'}$. Note that in $G_{\mathcal{A},\pi,\pi'}$ the edges are oriented. Some edges are oriented from the vertices in A_π to the vertices in $A_{\pi'}$, and some from the vertices in $A_{\pi'}$ to the vertices in A_π. It is complete since for every pair $(u, v) \in A_\pi \times A_{\pi'}$, either (u, v) or (v, u) is a μ_i-edge, for some μ_i.

See Figure 1 for an illustration of a graph representation of a structure $\mathcal{A}$ with domain $\{u_1, u_2, u_3, v_1, v_2, w\}$. The 1-types are π_1, π_2, π_3, and 2-types are $\mu_1, \mu_2, \mu_3, \mu_4$. In the graph representation the edge between u_1 and v_1 is oriented from u_1 to v_1 and the 2-type of (u_1, v_1) is μ_1.

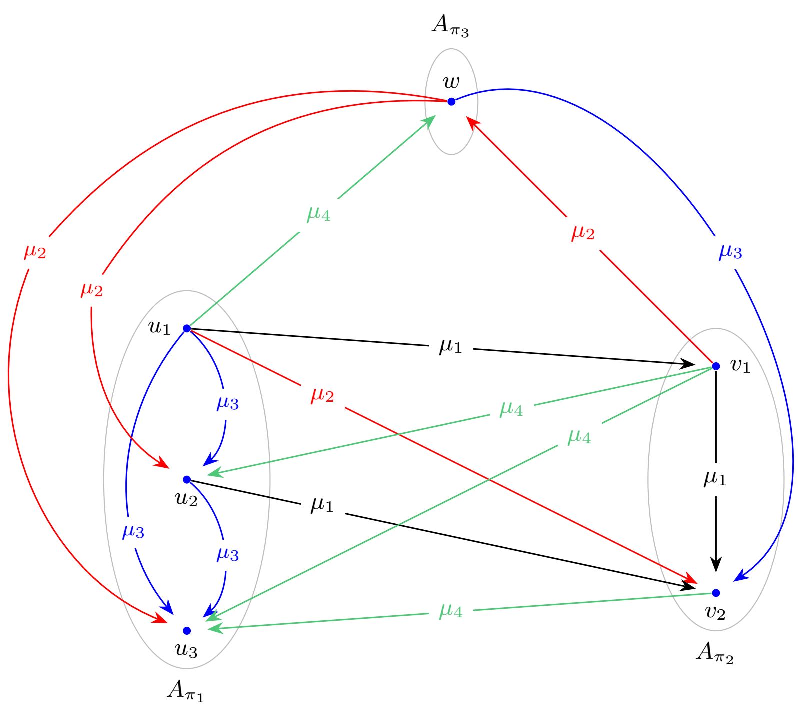

FIG. 1. *Illustration of a graph representation of a structure with 1-types π_1, π_2, π_3. The 2-types are $\mu_1, \mu_2, \mu_3, \mu_4$ represented by edges with color black, red, blue, and green, respectively. The vertices u_1, u_2, u_3 are in A_{π_1}, v_1, v_2 are in A_{π_2}, and w is in A_{π_3}. Note: color appears only in the online article.*

Remark 3.9. As we will see later, Theorem 3.3 can be used to characterize the size of the subgraph $G_{\mathcal{A},\pi}$. However, to use Theorem 3.2 to characterize the size of the subgraph $G_{\mathcal{A},\pi,\pi'}$, we need to view $G_{\mathcal{A},\pi,\pi'}$ as a $2t$-color complete *undirected* bipartite graph where the first t colors are used to represent the edges that are oriented from left to right. and the next t colors are used to represent the edges that are oriented from right to left.

Remark 3.10. It is worth noting that for a structure $\mathcal{A}$, the graph representation of $\mathcal{A}$ is not unique since it depends on the orientation put between the vertices. On the other hand, a graph representation uniquely defines a structure since the information about the vertices and the edges in a graph representation, i.e., the 1- and 2-types, uniquely determines the relations in the structure.

The biregular graph problem which our reduction produces will involve counting the possible sizes of certain partitions in the vector of graphs $G_{\mathcal{A},\pi,\pi'}$ and $G_{\mathcal{A},\pi}$, for every graph representation $G_{\mathcal{A}}$ of every structure $\mathcal{A} \models \phi$. We now explain the partitions we are looking for.

Let $g : \{\text{out}, \text{in}\} \times \mathcal{E} \times \Pi \to \mathbb{N}_{\infty,+p}$ be a function. We will use g to describe the "behavior" of elements in a graph representation $G_{\mathcal{A}}$ in the following sense. We say that an element $u \in A$ *behaves according to* g in a graph representation $G_{\mathcal{A}}$ if, for every $\pi \in \Pi$ and for every $\mu \in \mathcal{E}$,

- the number of outgoing μ-edges in the graph $G_{\mathcal{A}}$ from u to vertices in A_π is $g(\text{out}, \mu, \pi)$;
- the number of incoming μ-edges in the graph $G_{\mathcal{A}}$ to u from vertices in A_π is $g(\text{in}, \mu, \pi)$.

For example, in the graph representation in Figure 1 the element w behaves according to the following function g_1:

- $g_1(\text{out}, \mu_2, \pi_1) = 2$, $g_1(\text{out}, \mu_3, \pi_2) = 1$, $g_1(\text{in}, \mu_2, \pi_2) = 1$, $g_1(\text{in}, \mu_4, \pi_1) = 1$;
- g_1 maps all the other tuples in $\{\text{out}, \text{in}\} \times \mathcal{E} \times \Pi$ to 0.

As another example, the element u_1 behaves according to the following function g_2:

- $g_2(\text{out}, \mu_1, \pi_2) = 1$, $g_2(\text{out}, \mu_2, \pi_2) = 1$, $g_2(\text{out}, \mu_3, \pi_1) = 2$. And $g_2(\text{out}, \mu_4, \pi_3) = 1$;.
- the rest are mapped to 0.

We will call a function $g : \{\text{out}, \text{in}\} \times \mathcal{E} \times \Pi \to \mathbb{N}_{\infty,+p}$ a *behavior.* The restriction of g on 1-type π is the function $g_\pi : \{\text{out}, \text{in}\} \times \mathcal{E} \to \mathbb{N}_{\infty,+p}$, where $g_\pi(\kappa, \mu) = g(\kappa, \mu, \pi)$ for every $\kappa \in \{\text{out}, \text{in}\}$ and $\mu \in \mathcal{E}$. We call the function g_π the behavior (function) towards 1-type π.

We are, of course, only interested in 1-types and behaviors that are "allowed" by the sentence ϕ we are considering. To formalize this, we will use the following terminology, where $\alpha(x,y)$, $\beta_i(x,y)$, and S_i are from the fixed ϕ.

- A 1-type $\pi \in \Pi$ is *compatible* (with ϕ) if[3] $\pi(x) \models \alpha(x,x)$. Otherwise, we say that π is *incompatible.* Intuitively, π is incompatible means that whenever $\mathcal{A} \models \phi$, there is no element with 1-type π.
- For a 1-type $\pi \in \Pi$, for a behavior function $g : \{\text{out}, \text{in}\} \times \mathcal{E} \times \Pi \to \mathbb{N}_{\infty,+p}$, we say that (π, g) is compatible (with ϕ) if, for every $\mu \in \mathcal{E}$ and for every $\pi' \in \Pi$, If $g(\text{out}, \mu, \pi') \neq 0$, then

$$\pi(x) \wedge \mu(x,y) \wedge \pi'(y) \models \alpha(x,y) \quad \text{and} \quad \pi(y) \wedge \mu(y,x) \wedge \pi'(x) \models \alpha(x,y)$$

and if $g(\text{in}, \mu, \pi') \neq 0$, then

[3]As usual, we use $\models$ in both $\mathcal{A} \models \phi$ (for "$\mathcal{A}$ satisfies ϕ") and $\phi_1 \models \phi_2$ (for "ϕ_1 implies ϕ_2").

$$\pi(x) \wedge \mu(y,x) \wedge \pi'(y) \models \alpha(x,y) \quad \text{and} \quad \pi(y) \wedge \mu(x,y) \wedge \pi'(x) \models \alpha(x,y).$$

Otherwise, we say that (π, g) is incompatible. Intuitively, (π, g) is incompatible means that whenever $\mathcal{A} \models \phi$, there is no element in A_π that behaves according to g in any graph representation $G_{\mathcal{A}}$ of $\mathcal{A}$.

- A function g is a *good* behavior (w.r.t. ϕ) if for every $i \in [k]$:[4]

$$\sum_{\mu \ni \beta_i(x,y)} \sum_{\pi \in \Pi} g(\text{out}, \mu, \pi) \; + \; \sum_{\mu \ni \beta_i(y,x)} \sum_{\pi \in \Pi} g(\text{in}, \mu, \pi) \;\; \in \;\; S_i. \tag{3.2}$$

Intuitively, for a vertex u in a graph representation $G_{\mathcal{A}}$ that behaves according to g, the sum $\sum_{\mu \ni \beta_i(x,y)} \sum_{\pi \in \Pi} g(,^{\text{out}} \mu, \pi)$ is the number of outgoing edges that contains the relation $\beta_i(x,y)$ and the sum $\sum_{\mu \ni \beta_i(y,x)} \sum_{\pi \in \Pi} g(\text{in}, \mu, \pi)$ is the number of incoming edges that contains the relation $\beta_i(y,x)$. Their total sum is the number of elements v such that $\mathcal{A}, x/u, y/v \models \beta_i(x,y)$. Hence, when $\mathcal{A} \models \phi$, it must be inside the set S_i.

The notion of compatibility will be used to capture the universal part $\forall x \forall y \alpha(x,y)$ of our formula. The notion of good function will be used to capture the universally-quantified and Presburger-quantified part: $\bigwedge_{i=1}^{k} \forall x \exists y^{S_i} \beta_i(x,y) \wedge x \neq y$.

We observe that, for every structure $\mathcal{A} \models \phi$, for every graph representation $G_{\mathcal{A}}$ of $\mathcal{A}$, each vertex in $G_{\mathcal{A}}$ behaves according to a function g where the range is a subset of $\{0, \ldots, q, 0^{+p}, \ldots, q^{+p}, \infty\}$ for q the maximal non-∞ offset in all S_i (when seen as finite sets of linear sets). Indeed, suppose $\mathcal{A} \models \phi$ and let $G_{\mathcal{A}}$ be its graph representation. Let u be an element that behaves according to g. Suppose $g(\text{out}, \mu, \pi) = a$ or a^{+p} for some $a > q$, $\mu \in \mathcal{E}$, and $\pi \in \Pi$. We will show that u also behaves according to a function g' where g' is the same function as g except that $g'(\text{out}, \mu, \pi)$ is now $(a - sp)^{+p}$, where s is the minimum integer such that $a - sp \leqslant q$. We consider the case where $g(\text{out}, \mu, \pi) = a$. Suppose $\beta_i(x,y) \in \mu$, where $i \in [k]$. Let b denote the number of elements v such that $\mathcal{A}, x/u, y/v \models \beta_i(x,y) \wedge x \neq y$. Since u behaves according to g, we have $b \geqslant a$ and hence $b > q$. Moreover, $b \in S_i$ since $\mathcal{A} \models \phi$. Because $b > q$ there must be $c^{+p} \in S_i$ such that $b \in c^{+p}$. This means that u also behaves according to g' where g' is the same as g except that $g'(\text{out}, \mu, \pi) = (a - sp)^{+p}$, where s is the minimum integer such that $a - sp \leqslant q$. The cases where $g(\text{out}, \mu, \pi) = a^{+p}$ or $g(\text{in}, \mu, \pi) = a$ or $g(\text{in}, \mu, \pi) = a^{+p}$ with $a > q$ can be treated in a similar manner.

So we may concentrate on only the good behaviors whose codomain is $\{0, \ldots, q, 0^{+p}, \ldots, q^{+p}, \infty\}$, where q is the maximal non-∞ offset in all S_i. Below we will partition elements based on their behaviors, always using good behaviors, thus the partitions will be finite.

For $\mathcal{A} \models \phi$, and for a graph representation $G_{\mathcal{A}}$ of $\mathcal{A}$, we can partition $A = A_{\pi_1, g_1} \uplus \cdots \uplus A_{\pi_n, g_m}$ according to the 1-types and good behavior functions: for every element $u \in A$, we pick a behavior function g_j such that u behaves according to g (in $G_{\mathcal{A}}$), and declare that $u \in A_{\pi_i, g_j}$ where π_i is the 1-type of u.[5] We can then consider the vector of subgraphs $G_{\mathcal{A},\pi}$ and $G_{\mathcal{A},\pi,\pi'}$ of $G_{\mathcal{A}}$. We call this the *type-behavior partitioned graph vector* associated with the graph $G_{\mathcal{A}}$. The term "partitioned" refers to the fact that the vertices of $G_{\mathcal{A},\pi}$ have a natural partition into $A_{\pi,g}$ for differing g. Intuitively, to decide whether ϕ is satisfiable, we construct a Presburger formula

[4] Here the operation $+$ on $\mathbb{N}_{\infty,+p}$ is defined to be the commutative extension of the standard addition on $\mathbb{N}$ such that $a + \infty = a^{+p} + \infty = \infty$ and $a^{+p} + b = a^{+p} + b^{+p} = (a+b)^{+p}$.

[5] In general, for an element $u \in A$, there may be several behaviors according to which u behaves; we partition the domain by picking one such behavior.

that captures the sizes of all the subgraphs in the type-behavior partitioned graph vector associated by the graph $G_{\mathcal{A}}$, for every possible graph representation $G_{\mathcal{A}}$ of every model $\mathcal{A} \models \phi$.

At this point we can expand on the intuition for reducing satisfiability to biregular graph problems. We will construct a sentence PRES_ϕ that "counts" the possible cardinalities of the subgraphs in a type-behavior partitioned graph vector associated with a graph representation $G_{\mathcal{A}}$ for a model $\mathcal{A}$ of ϕ.

Recall that $\Pi = \{\pi_1, \pi_2, \ldots, \pi_n\}$ is the set of all 1-types. Let $\mathcal{G} = \{g_1, \ldots, g_m\}$ be the set of all behavior functions whose codomain is $\{0, \ldots, q, 0^{+p}, \ldots, q^{+p}, \infty\}$. The sentence PRES_ϕ will be of the form

$$\mathsf{PRES}_\phi := \exists \bar{X}\ \mathsf{consistent}_1(\bar{X}) \wedge \mathsf{consistent}_2(\bar{X}) \wedge \left(\bigvee_{i \in [n],\ j \in [m]} X_{\pi_i, g_j} \neq 0 \right), \tag{3.3}$$

where $\bar{X}$ is a vector of variables $(X_{\pi_1,g_1}, X_{\pi_1,g_2}, \ldots, X_{\pi_n,g_m})$. Intuitively, each X_{π_i,g_j} represents $|A_{\pi_i,g_j}|$ in some graph representation G. The final conjunct ensures that the domain is nonempty. By the formula $\mathsf{consistent}_1(\bar{X})$, we capture the consistency of the nonnegative integers $\bar{X}$ with the first conjunct $\forall x \forall y\ \alpha(x,y)$ of ϕ. By the formula $\mathsf{consistent}_2(\bar{X})$, we capture the consistency of the nonnegative integers $\bar{X}$ with the second conjuncts $\bigwedge_{i=1}^{k} \forall x \exists^{S_i} y\ \beta_i(x,y) \wedge x \neq y$. In $\mathsf{consistent}_2(\bar{X})$ we will consider the type-behavior partitioned graph vector as the common solution of a set biregular graph and digraph problems, and make use of the Presburger definability of biregular graph problems.

Towards defining the formulas $\mathsf{consistent}_1$ and $\mathsf{consistent}_2$, we define matrices that will constrain the partitions.

$$M_\pi^{\mathrm{out}} := \begin{pmatrix} g_1(\mathrm{out}, \mu_1, \pi) & \cdots & g_m(\mathrm{out}, \mu_1, \pi) \\ \vdots & \ddots & \vdots \\ g_1(\mathrm{out}, \mu_t, \pi) & \cdots & g_m(\mathrm{out}, \mu_t, \pi) \end{pmatrix}$$

and

$$M_\pi^{\mathrm{in}} := \begin{pmatrix} g_1(\mathrm{in}, \mu_1, \pi) & \cdots & g_m(\mathrm{in}, \mu_1, \pi) \\ \vdots & \ddots & \vdots \\ g_1(\mathrm{in}, \mu_t, \pi) & \cdots & g_m(\mathrm{in}, \mu_t, \pi) \end{pmatrix}.$$

That is, M_π^{out} contains information about the outgoing edges toward 1-type π and M_π^{in} contains information about the incoming edges from 1-type π.

Now, we explain how to capture information about the relationship between elements with distinct 1-types. Define matrices $L_\pi, L_\pi^{\mathrm{rev}} \in \mathbb{N}_{\infty,+p}^{2t \times m}$,

$$L_\pi := \begin{pmatrix} M_\pi^{\mathrm{out}} \\ M_\pi^{\mathrm{in}} \end{pmatrix} \quad \text{and} \quad L_\pi^{\mathrm{rev}} := \begin{pmatrix} M_\pi^{\mathrm{in}} \\ M_\pi^{\mathrm{out}} \end{pmatrix}; \tag{3.4}$$

that is, in L_π the first t rows, corresponding to t edge colors, come from M_π^{out} with the next t rows from M_π^{in}. While in L_π^{rev} the first t rows come from M_π^{in}, followed by the t rows from M_π^{out}.

The intended meaning of the matrices is as follows. For every structure $\mathcal{A}$, for every graph representation $G_{\mathcal{A}}$ of $\mathcal{A}$, $\mathcal{A} \models \phi$ if and only

- for every 1-type π, the subgraph $G_{\mathcal{A},\pi}$ is a complete $M_\pi^{\mathrm{out}} | M_\pi^{\mathrm{in}}$-regular digraph;

- for distinct 1-types π, π', the subgraph $G_{\mathcal{A},\pi,\pi'}$ is a complete $L_{\pi'}|L_{\pi}^{\text{rev}}$-biregular graph.
  Here the first t rows in $L_{\pi'}|L_{\pi}^{\text{rev}}$ capture the edges in $G_{\mathcal{A},\pi,\pi'}$ that are oriented from left to right, whereas the last t rows capture the edges in $G_{\mathcal{A},\pi,\pi'}$ that are oriented from right to left.

We are now ready to define the formulas, beginning with $\mathsf{consistent}_1(\bar{X})$. Letting H be the set of all incompatible pairs (π, g), the formula $\mathsf{consistent}_1(\bar{X})$ can be defined as follows:

$$\begin{aligned}\mathsf{consistent}_1(\bar{X}) := & \bigwedge_{\pi \text{ is incompatible, } g\in\mathcal{G}} X_{\pi,g} = 0 \quad \wedge \bigwedge_{(\pi,g)\in H} X_{\pi,g} = 0 \\ & \wedge \bigwedge_{g \text{ is not a good function, } \pi\in\Pi} X_{\pi,g} = 0.\end{aligned}$$

We turn to formula $\mathsf{consistent}_2(\bar{X})$. Recall that we enumerated all the 1-types as $\pi_1, \ldots, \pi_n$. We now define $\mathsf{consistent}_2$, where below each $\bar{X}_{\pi_i}$ is the vector $(X_{\pi_i,g_1}, X_{\pi_i,g_2}, \ldots, X_{\pi_i,g_m})$ and each $\bar{X}_{\pi_j}$ is defined in the same way:

$$\begin{aligned}&\mathsf{consistent}_2(\bar{X}) \\ &:= \bigwedge_{1\leqslant i<j\leqslant n} \mathsf{c\text{-}bireg}_{L_{\pi_j}|L_{\pi_i}^{\text{rev}}}(\bar{X}_{\pi_i}, \bar{X}_{\pi_j}) \ \wedge \bigwedge_{1\leqslant i\leqslant n} \mathsf{c\text{-}reg}_{M_{\pi_i}^{\text{out}}|M_{\pi_i}^{\text{in}}}(\bar{X}_{\pi_i}).\end{aligned} \tag{3.5}$$

Observe that formula $\mathsf{consistent}_1(\bar{X})$ is Presburger definable by inspection, while $\mathsf{consistent}_2(\bar{X})$ is Presburger definable using Theorems 3.2 and 3.3. Thus, the sentence PRES_ϕ is an existential Presburger sentence. Lemmas 3.11 and 3.12 show that PRES_ϕ is indeed the sentence required by Theorem 3.6.

LEMMA 3.11. *For each structure $\mathcal{A} \models \phi$, for every graph representation $G_\mathcal{A}$ of $\mathcal{A}$, there is a partition $A = A_{\pi_1,g_1} \uplus \cdots \uplus A_{\pi_n,g_m}$ such that*

- *for every $\pi_i \in \Pi$, for every $g_j \in \mathcal{G}$, A_{π_i,g_j} contains the elements with 1-type π_i and behaves according to g_j in the graph representation $G_\mathcal{A}$;*
- *the subgraphs in the type-behavior partitioned graph vector associated with $G_\mathcal{A}$ are complete regular and biregular graphs in the following sense:*
  (a) *For every $\pi_i \in \Pi$, $G_{\mathcal{A},\pi_i}$ is a complete $M_{\pi_i}^{out}|M_{\pi_i}^{in}$-regular digraph with witness partition $A_{\pi_i} = A_{\pi_i,g_1} \uplus \cdots \uplus A_{\pi_i,g_m}$.*
  (b) *For every $\pi_i, \pi_j \in \Pi$, where $i \neq j$, $G_{\mathcal{A},\pi_i,\pi_j}$ is a complete $L_{\pi_j}|L_{\pi_i}^{rev}$-biregular graph with witness partition $A_{\pi_i} = A_{\pi_i,g_1} \uplus \cdots \uplus A_{\pi_i,g_m}$ and $A_{\pi_j} = A_{\pi_j,g_1} \uplus \cdots \uplus A_{\pi_j,g_m}$;*
- $\mathsf{consistent}_1(\bar{N}) \ \wedge\ \mathsf{consistent}_2(\bar{N}) \ \wedge\ \bigvee_{i\in[n],\ j\in[m]} |A_{\pi_i,g_j}| \neq 0$ *holds in $\mathcal{N}_\infty$, where $\bar{N} = (|A_{\pi_1,g_1}|, \ldots, |A_{\pi_n,g_m}|)$.*

Proof. Let $\mathcal{A} \models \phi$. We fix a graph representation $G_\mathcal{A}$. We partition A into $A_{\pi_1,g_1} \uplus \cdots \uplus A_{\pi_n,g_m}$, where for every element $u \in A$, we pick a behavior function g_j such that u behaves according to g (in $G_\mathcal{A}$), and declare that $u \in A_{\pi_i,g_j}$ where π_i is the 1-type of u. Obviously, the first bullet item holds.

To prove item (a) in the second bullet item, let $\pi_i \in \Pi$. By construction, for every $g_k \in \mathcal{G}$, every element in A_{π_i,g_k} behaves according to g_k. Moreover, $G_{\mathcal{A},\pi_i}$ is a complete digraph. Thus, by the definition of $M_{\pi_i}^{\text{out}}$ and $M_{\pi_i}^{\text{in}}$, the subgraph $G_{\mathcal{A},\pi_i}$ is a complete $M_{\pi_i}^{\text{out}}|M_{\pi_i}^{\text{in}}$-regular digraph with witness partition $A_{\pi_i} = A_{\pi_i,g_1} \uplus \cdots \uplus A_{\pi_i,g_m}$. Item (b) in the second bullet item can be proved in a similar manner.

We now prove the third bullet item. Since $\mathcal{A}$ contains at least one element, at least one of the $A_{\pi,g}$'s is not empty. Hence the last conjunct $\bigvee_{i\in[n],\ j\in[m]} |A_{\pi_i,g_j}| \neq 0$

holds. Since $\mathcal{A} \models \forall x \forall y\ \alpha(x,y)$, $A_{\pi_i} = \emptyset$ whenever π_i is incompatible and $A_{\pi_i,g_j} = \emptyset$ whenever (π_i, g_j) is incompatible, the following conjunct holds.

$$\bigwedge_{\pi \text{ is incompatible, } g\in\mathcal{G}} |A_{\pi,g}| = 0 \qquad \wedge \qquad \bigwedge_{(\pi,g)\in H} |A_{\pi,g}| = 0.$$

Moreover, since $\mathcal{A} \models \bigwedge_{i=1}^{k} \forall x \exists^{S_i} y\ \beta_i(x,y) \wedge x \neq y$, the following conjunct holds:

$$\bigwedge_{g \text{ is not a good function, } \pi\in\Pi} |A_{\pi,g}| = 0.$$

Thus, $\mathsf{consistent}_1(\bar{N})$ holds for the assignment. Finally, $\mathsf{consistent}_2(\bar{X})$ holds due to bullet item 2 and Theorems 3.3 and 3.2. ∎

Next, we prove the converse direction of Lemma 3.11.

LEMMA 3.12. *For every nonzero vector $\bar{N}$ such that $\mathsf{consistent}_1(\bar{N}) \wedge \mathsf{consistent}_2(\bar{N})$ holds in $\mathcal{N}_\infty$, there is a structure $\mathcal{A} \models \phi$, a graph representation $G_{\mathcal{A}}$, and a partition $A = A_{\pi_1,g_1} \uplus \cdots \uplus A_{\pi_n,g_m}$ such that*

- *$\bar{N} = (|A_{\pi_1,g_1}|, \ldots, |A_{\pi_n,g_m}|)$;*
- *for every $\pi_i \in \Pi$, for every $g_j \in \mathcal{G}$, A_{π_i,g_j} contains the elements with 1-type π_i and behaves according to g_j in the graph representation $G_{\mathcal{A}}$;*
- *the subgraphs in the type-behavior partitioned graph vector associated with $G_{\mathcal{A}}$ are complete regular and biregular graphs in the sense of (a) and (b) in Lemma 3.11 above.*

Proof. Let $\bar{N} = (N_{\pi_1,g_1}, \ldots, N_{\pi_n,g_m})$ be a nonzero vector such that $\mathsf{consistent}_1(\bar{N}) \wedge \mathsf{consistent}_2(\bar{N})$ holds. For each $i \in [n]$, let $\bar{N}_{\pi_i} = (N_{\pi_i,g_1}, \ldots, N_{\pi_i,g_m})$.

For each $(\pi_i, g_j) \in \Pi \times \mathcal{G}$, we have a set V_{π_i,g_j} with cardinality N_{π_i,g_j}. We let $V_{\pi_i} = \bigcup_{g_j\in\mathcal{G}} V_{\pi_i,g_j}$ for each $\pi_i \in \Pi$. We construct a structure $\mathcal{A} \models \phi$ along with a particular graph representation $G_{\mathcal{A}}$.

- The domain is $A = \bigcup_{\pi_i\in\Pi,\ g_j\in\mathcal{G}} V_{\pi_i,g_j}$.
  Note that since $\bar{N}$ is a nonzero vector, at least one V_{π_i,g_j} is not empty and, therefore, A is not empty.
- For each $\pi_i \in \Pi$ and for each element $u \in V_{\pi_i}$, the predicates that hold on u are defined such that the 1-type of u is π_i.
- For each $\pi_i \in \Pi$, we define the 2-types of each pair $(u,v) \in V_{\pi_i} \times V_{\pi_i}$, where $u \neq v$ as follows.
  Since $\mathsf{c\text{-}reg}_{M^{\mathrm{out}}_{\pi_i}|M^{\mathrm{in}}_{\pi_i}}(\bar{N}_{\pi_i})$ holds, by Theorem 3.3, there is a complete $M^{\mathrm{out}}_{\pi_i}|M^{\mathrm{in}}_{\pi_i}$-regular digraph $G_{\pi_i} = (V, E_1, \ldots, E_t)$ with size $\bar{N}_{\pi_i}$. Note that we can take the set V_{π_i} as the domain V of the graph and $V = V_{\pi_i,g_1} \uplus \cdots \uplus V_{\pi_i,g_m}$ as the witness partition since $(|V_{\pi_i,g_1}|, \ldots, |V_{\pi_i,g_m}|) = \bar{N}_{\pi_i}$ by construction. Then, for every $1 \leqslant j \leqslant t$, we set the 2-types of the edges in E_j as μ_j. We define the subgraph $G_{\mathcal{A},\pi_i}$ as the graph G_{π_i} itself.
- For every $\pi_i, \pi_j \in \Pi$ with $i < j$, we now define the 2-types of each pair $(u,v) \in V_{\pi_i} \times V_{\pi_j}$.
  Since $\mathsf{c\text{-}bireg}_{L_{\pi_j}|L^{\mathrm{rev}}_{\pi_i}}(\bar{N}_{\pi_i}, \bar{N}_{\pi_j})$ holds, applying Theorem 3.2, there is a complete $L_{\pi_j}|L^{\mathrm{rev}}_{\pi_i}$-biregular graph $G_{\pi_i,\pi_j} = (V_{\pi_i}, V_{\pi_j}, E_1, \ldots, E_t, E_{t+1}, \ldots, E_{2t})$ with size $\bar{N}_{\pi_i}|\bar{N}_{\pi_j}$.
  Again, note that we can take V_{π_i} and V_{π_j} as the set of vertices on the left-hand side and the right-hand side of the graph G_{π_i,π_j}, respectively, and that $V_{\pi_i} = V_{\pi_i,g_1} \uplus \cdots \uplus V_{\pi_i,g_m}$ and $V_{\pi_j} = V_{\pi_j,g_1} \uplus \cdots \uplus V_{\pi_j,g_m}$ as the witness

partition of $L_{pij}|L_{pii}^{rev}$-biregularity since the sizes $(|V_{\pi_i,g_1}|,\ldots,|V_{\pi_i,g_m}|)$ and $(|V_{\pi_j,g_1}|,\ldots,|V_{\pi_j,g_m}|)$ match the vectors $\bar{N}_{\pi_i}$ and $\bar{N}_{\pi_j}$ by construction. We set the 2-types of each pair $(u,v)\in V_{\pi_i}\times V_{\pi_j}$ as follows.

– If $(u,v)\in E_h$, for some $1\leqslant h\leqslant t$, then we set the 2-type of (u,v) to be μ_h.

– If $(u,v)\in E_h$, for some $t+1\leqslant h\leqslant 2t$, then we set the 2-type of (v,u) to be μ_h.

We define the subgraph $G_{\mathcal{A},\pi_i,\pi_j}=(V_{\pi_i},V_{\pi_j},E'_1,\ldots,E'_t)$, where for each $1\leqslant h\leqslant t$, $E'_h=E_h\cup\{(v,u)\mid(u,v)\in E_{h+t}\}$, i.e., we treat the edges in E_h and E_{h+t} as having the same color, but the orientation of the edges in E_h is from left to right and the orientation of the edges in E_h is from right to left.

The above process produces a model $\mathcal{A}$ and a graph representation $G_{\mathcal{A}}$ as well as the type-behavior partitioned graph vector. It is easy to see that for every 1-type π_i, every behavior function g_j, every vertex $u\in V_{\pi_i,g_j}$ has 1-type π_i and behaves according to the function g_j.

To show that $\mathcal{A}\models\phi$, we first show that $\mathcal{A}\models\forall x\forall y\ \alpha(x,y)$. Let $u,v\in A$. There are two cases.

- When $u=v$ and $u\in V_\pi$. This means $V_\pi\neq\emptyset$. Hence $|V_\pi|=\sum_{g\in\mathcal{G}}N_{\pi,g}\neq 0$. Therefore, π is compatible, which by definition means $\pi(x)\models\alpha(x,x)$. By the construction of $\mathcal{A}$, we have $\mathcal{A},x/u,y/u\models\alpha(x,y)$.
- When $u\neq v$ and $u\in V_\pi$ and $v\in V_{\pi'}$. Suppose the 2-type of (u,v) is μ and the orientation is from u to v in the graph $G_{\mathcal{A}}$. This means there is $g\in\mathcal{G}$ such that $g(\text{out},\mu,\pi')\neq 0$ and $u\in V_{\pi,g}$, which implies that $V_{\pi,g}\neq\emptyset$, i.e., $N_{\pi,g}\neq 0$. Since $\mathsf{consistent}_1(\bar{N})$ holds, which states that $N_{\pi,g}=0$ whenever (π,g) is incompatible, the pair (π,g) is compatible:
$$\pi(x)\wedge\mu(x,y)\wedge\pi'(y)\ \models\ \alpha(x,y)\quad\text{ and }\quad\pi(y)\wedge\mu(y,x)\wedge\pi'(x)\ \models\ \alpha(x,y).$$
Since $\mathcal{A}$ is a structure with graph representation $G_{\mathcal{A}}$, we have $\mathcal{A},x/u,y/v\models\alpha(x,y)$ and $\mathcal{A},x/v,y/u\models\alpha(x,y)$. The case when the orientation is from v to u can be treated in a similar manner.

Next, we show that $\mathcal{A}\models\bigwedge_{i=1}^{k}\forall x\exists^{S_i}y\ \beta_i(x,y)\wedge x\neq y$. Fix $u\in A$. Let $\pi\in\Pi$ and $g\in\mathcal{G}$ such that $u\in V_{\pi,g}$, i.e., u behaves according to $g\in\mathcal{G}$ in the graph $G_{\mathcal{A}}$. Thus, $V_{\pi,g}\neq\emptyset$. Since $\mathsf{consistent}_1(\bar{N})$ holds, $|V_{\pi,g}|=N_{\pi,g}$ and $N_{\pi,g}\neq 0$: the function g is good. By the construction of the graph $G_{\mathcal{A}}$, for every $i\in[k]$, the number of elements $y\neq u$ such that $\beta_i(x,y)$ belongs to the 2-type of (u,y) is the sum
$$\sum_{\mu\ni\beta_i(x,y)}\ \sum_{\pi'\in\Pi}g(\text{out},\mu,\pi')\ +\ \sum_{\mu\ni\beta_i(y,x)}\ \sum_{\pi'\in\Pi}g(\text{in},\mu,\pi'). \tag{3.6}$$
By the definition of a good function, for every $i\in[k]$, the sum (3.6) is an element in S_i. Therefore, $\mathcal{A},x/u\models\exists^{S_i}y\ \beta_i(x,y)\wedge x\neq y$ for every $i\in[k]$. Since the choice of u is arbitrary, $\mathcal{A}\models\forall x\exists^{S_i}y\ \beta_i(x,y)\wedge x\neq y$. ◻

Thus, we have shown that, for every $\mathsf{FO}^2_{\text{Pres}}$ sentence ϕ in normal form (3.1), we can effectively construct an existential Presburger sentence $\mathsf{PRES}^{\infty}_{\phi}$ such that ϕ has a model if and only if $\mathsf{PRES}^{\infty}_{\phi}$ holds in $\mathcal{N}_\infty$. By Remark 3.5, the formula $\mathsf{PRES}^{\infty}_{\phi}$ can be easily rewritten to another formulaPRES_{ϕ} such that ϕ has a finite model if and only if PRES_{ϕ} holds in $\mathcal{N}$. The sentences $\mathsf{PRES}^{\infty}_{\phi}$ and PRES_{ϕ} are as required by Theorem 3.6.

Remark 3.13. Note that in a type-behavior partitioned graph vector, information about 2-types is coded in both the edge relation and in the partition, since the partition is defined via behavior functions. Thus there are additional dependencies on sizes for a type-behavior partitioned graph vector of a model of ϕ, beyond what will be captured in outdegree constraints.

This will not be a problem for us, because these dependencies could be captured by additional Presburger constraints. We highlight that to solve satisfiability for our logic, it was not sufficient to know whether a biregular graph problem is solvable: we needed to get a Presburger formula for the possible cardinalities, which we combine with these additional constraints.

4. Proof ideas using a special case for the graph analysis results (Theorems 3.2 and 3.3). We now discuss the proofs of the main (bi)regular graph theorems. These theorems deal with matrices that may contain infinite entries, as well as matrices that can contain periodic entries. Thus elements of the witness partitions can be forced to be infinite or finite. *In the body of the paper we restrict our analysis to graphs that are finite, and thus in particular ignore the possibility of an infinite entry.* This suffices to show the claimed bounds on the finite satisfiability problem for our logic. In the appendix we explain the extensions needed to deal with the infinite case, and thus the general satisfiability problem.

We start in this section by giving proofs only for the 1-*color case,* without the completeness requirement. While this case does not directly correspond to any formula used in the proof of Theorem 3.6 (since matrices (3.4) have 2 rows even when there are no binary predicates), this case gives the flavor of the arguments, and will also be used as the base cases in inductive constructions for the case with arbitrary colors. This will be bootstrapped to the multicolor case in later sections. Note that the 1-color case *with* the completeness requirement is not very interesting, and also not useful for the general case: completeness states that every node on the left must be connected, via the unique edge relation, to every node on the right—regardless of the matrix. We can easily write down equations that capture this.

This section is organized as follows. In subsection 4.1 we will focus on the version of Theorem 3.2 for 1-color biregular graphs. In subsection 4.2 we present a brief explanation of how to modify the proof for regular digraphs (i.e., the case of Theorem 3.3). In this section and also in the next, we will be concerned with effectiveness but not complexity. The complexity of our procedures will be analyzed in section 8.

4.1. The case of incomplete 1-color biregular graphs. We will begin by proving a result for 1-color biregular graphs without the completeness requirement.

LEMMA 4.1. *For every pair of degree matrices $A \in \mathbb{N}_{+p}^{1\times m}$ and $B \in \mathbb{N}_{+p}^{1\times n}$, there exists an (effectively computable) existential Presburger formula $\mathsf{bireg}_{A|B}(\bar{x},\bar{y})$ such that for every pair of size vectors $\bar{M} \in \mathbb{N}^m$ and $\bar{N} \in \mathbb{N}^n$, the formula $\mathsf{bireg}_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}$ if and only if there is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$.*

Our strategy to prove Lemma 4.1 is to divide it into two main cases. The first case deals with the graphs with "big-enough" sizes and the second case with the graphs with "not-big-enough" sizes. We organize the rest of section 4.1 as follows. In section 4.1.1 we introduce some notation and the formal definition of big-enough sizes. Then, in section 4.1.2, we present the formula that captures $A|B$-biregular graphs with big-enough sizes. The not-big-enough sizes will be handled in section 4.1.3.

4.1.1. Notation and terminology. We will use the following notation. The term "vectors" always refers to row vectors (of finite length). We use $\bar{a}, \bar{b}, \bar{M}, \bar{N}, \ldots$ (possibly indexed) to denote such row vectors. For a vector $\bar{a}$, we denote by a_j the

jth entry in $\bar{a}$. We write $(\bar{a}, \bar{b})$ to denote the row vector obtained by concatenating $\bar{a}$ with $\bar{b}$. We use $\cdot$ to denote the standard dot product between two vectors. To avoid being repetitive, when vector operations such as dot products/additions/subtractions are performed, it is implicit that the vector lengths are the same.

We now fix notation for degree matrices. Recall that, in our case, degree matrices are matrices with entries from $\mathbb{N}_{+p}$, where p is a positive integer which is a common nonzero period in all the set S_i's in (3.1). Obviously, 1-row matrices can be viewed as row vectors. Entries of the form a^{+p} in a degree matrix are called *periodic* entries. Otherwise, they are called *fixed* entries.

We write offset(a^{+p}), for a periodic entry a^{+p}, to denote the offset value a. Note that this is consistent with the definition of offset of the corresponding linear set from section 2. We define offset(a) for an integer a to be a itself. The offset of a vector $\bar{a}$, denoted by offset$(\bar{a})$, is the row vector obtained by replacing every entry a_j with offset(a_j). Of course, if $\bar{a}$ does not contain any periodic entry, then offset$(\bar{a})$ is $\bar{a}$ itself.

In the 1-color case, matrices A and B for $A|B$-biregular graphs are in fact row vectors. So, we will often write these matrices as $\bar{a}$ and $\bar{b}$, respectively. To differentiate between vectors that are supposed to represent the degrees of vertices in a graph and vectors that are supposed to represent the sizes of a graph, we call the former *degree vectors* and the latter *size vectors*. We usually write $\bar{a}, \bar{b}, \ldots$ to denote degree vectors and $\bar{M}, \bar{N}, \ldots$ to denote size vectors. Note that degree vectors have entries from $\mathbb{N}_{+p}$, whereas size vectors have entries from $\mathbb{N}$.

For degree vectors $\bar{a}$ and $\bar{b}$ containing only fixed entries, we write $\delta(\bar{a}, \bar{b})$ to denote $\max(\bar{a}, \bar{b})$, i.e., the maximal element in $\bar{a}$ and $\bar{b}$. When at least one of $\bar{a}$ and $\bar{b}$ contain periodic entries, we define $\delta(\bar{a}, \bar{b})$ as the maximal entry in $(\text{offset}(\bar{a}), \text{offset}(\bar{b}), p)$. For example, if $\bar{a} = (3, 1)$ and $\bar{b} = (2^{+5}, 4)$, then $\delta(\bar{a}, \bar{b})$ is the maximal entry in $(3, 1, 2, 4, 5)$, which is 5.

Let $\bar{a}$ be a degree vector. We let nz$(\bar{a})$ denote the set of indices j, where a_j is not 0. We let per$(\bar{a})$ denote the set of indices j where a_j is a periodic entry.

For a size vector $\bar{M}$ of length m, let $\|\bar{M}^T\|$ denote the sum of all the entries in $\bar{M}$, i.e., $\sum_{j=1}^{m} M_j$, that is, the 1-norm of the column vector $\bar{M}^T$, where $\bar{M}^T$ denotes the transpose of $\bar{M}$. For a subset $X \subseteq [m]$, we write $\|\bar{M}^T\|_X = \sum_{j \in X} M_j$ (which includes the case $\|\bar{M}^T\|_\emptyset = 0$). In this section we will only use $\|\bar{M}^T\|_X$, where X is nz$(\bar{a})$ or per$(\bar{a})$, for some degree vector $\bar{a}$.

The intuition is that if G is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$, then the norm $\|\bar{M}^T\|_{\text{nz}(\bar{a})}$ denotes the number of vertices on the left of the graph with nonzero degree bound and $\|\bar{M}^T\|_{\text{per}(\bar{a})}$ denotes the number of vertices where the corresponding entry of $\bar{a}$ is periodic. The meaning of $\|\bar{N}^T\|_{\text{nz}(\bar{b})}$ and $\|\bar{N}^T\|_{\text{per}(\bar{b})}$ is analogous with respect to the vertices on the right.

We now introduce the notion of big-enough sizes, the intuitive meaning of which will become apparent later on.

Definition 4.2. *Let $\bar{a}$ and $\bar{b}$ be degree vectors and let $\bar{M}$ and $\bar{N}$ be size vectors with the same length as $\bar{a}$ and $\bar{b}$, respectively. We say that $\bar{M}|\bar{N}$* is big-enough for *$\bar{a}|\bar{b}$, if each of the following holds:*[6]

(a) $\max(\|\bar{M}^T\|_{\text{nz}(\bar{a})}, \|\bar{N}^T\|_{\text{nz}(\bar{b})}) \geqslant 2\delta(\bar{a}, \bar{b})^2 + 1$;
(b) $\|\bar{M}^T\|_{\text{per}(\bar{a})} = 0$ *or* $\|\bar{M}^T\|_{\text{per}(\bar{a})} \geqslant \delta(\bar{a}, \bar{b})^2 + 1$;
(c) $\|\bar{N}^T\|_{\text{per}(\bar{b})} = 0$ *or* $\|\bar{N}^T\|_{\text{per}(\bar{b})} \geqslant \delta(\bar{a}, \bar{b})^2 + 1$.

[6] $(\delta(\bar{a}, \bar{b}))^2$ is abbreviated $\delta(\bar{a}, \bar{b})^2$.

In the following, to avoid clutter, when we say that $\bar{M}|\bar{N}$ is big-enough for $\bar{a}|\bar{b}$, it is implicit that $\bar{M}$ has the same length as $\bar{a}$ and $\bar{N}$ has the same length as $\bar{b}$. As usual, when presenting a Presburger formula, we will write $\bar{x}, \bar{y}, \bar{z}, \ldots$ (possibly indexed) to denote vectors of variables, where x_j denotes the jth entry in $\bar{x}$. We will also use the notation $\|\bar{x}^T\|$ to denote the sum of all the variables in $\bar{x}$ and, similarly, use $\|\bar{x}^T\|_X$ to denote the sum $\sum_{j \in X} x_j$.

4.1.2. The formula for the case of big-enough sizes. Note that for the conditions (b) and (c) required in the definition of big-enough, there are two possible subcases: either the norm is 0 or at least as big as some threshold. There are altogether 4 possible scenarios and our formula for big-enough sizes will be a disjunction of 4 formulas, one for each scenario. By symmetry, it suffices to consider the following three of these scenarios for the sizes $\bar{M}|\bar{N}$ of $\bar{a}|\bar{b}$-biregular graphs:

(S1) $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} = \|\bar{N}^T\|_{\mathrm{per}(\bar{b})} = 0$ (i.e., there are only vertices with fixed degree);

(S2) $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} \neq 0$ and $\|\bar{N}^T\|_{\mathrm{per}(\bar{b})} = 0$ (i.e., there are vertices with periodic degrees on exactly one side);

(S3) $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} \neq 0$ and $\|\bar{N}^T\|_{\mathrm{per}(\bar{b})} \neq 0$ (i.e., there are vertices with periodic degrees on both sides).

The rest of this section is devoted to the formulas for each of the cases above.

The formula and argument for scenario (S1)*: Partition on one side, merge, and swap.* Consider the formula $\psi^1_{\bar{a}|\bar{b}}(\bar{x}, \bar{y})$ defined as follows:

$$\mathrm{offset}(\bar{a}) \cdot \bar{x} = \mathrm{offset}(\bar{b}) \cdot \bar{y} \quad \wedge \quad \|\bar{x}^T\|_{\mathrm{per}(\bar{a})} = \|\bar{y}^T\|_{\mathrm{per}(\bar{b})} = 0. \tag{4.1}$$

Note that the last conjunct simply states that the condition of (S1) holds. The first conjunct is something we will see often, an *edge counting equality,* saying that the number of outgoing edges from the left must equal the number of incoming edges on the right.

LEMMA 4.3. *For every pair of degree vectors $\bar{a}, \bar{b}$ and for every $\bar{M}|\bar{N}$ big-enough for $\bar{a}|\bar{b}$, the formula $\psi^1_{\bar{a}|\bar{b}}(\bar{M}, \bar{N})$ holds in $\mathcal{N}$ if and only if there is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$, where* (S1) *holds.*

Proof. Let $\bar{a}, \bar{b}$ be degree vectors and $\bar{M}|\bar{N}$ be size vectors big-enough for $\bar{a}|\bar{b}$.

For the "if" direction, note that if we have an $\bar{a}|\bar{b}$-biregular graph G with size $\bar{M}|\bar{N}$, where (S1) holds, the total number of edges (by counting the edges adjacent to the vertices on the left) must be $\mathrm{offset}(\bar{a}) \cdot \bar{M}$. Similarly by considering the vertices on the right, the total number of edges must be $\mathrm{offset}(\bar{b}) \cdot \bar{N}$. Thus the condition $\mathrm{offset}(\bar{a}) \cdot \bar{M} = \mathrm{offset}(\bar{b}) \cdot \bar{N}$ is always a necessary one, regardless of whether $\bar{M}|\bar{N}$ is big-enough. Since the second conjunct of $\psi^1_{\bar{a}|\bar{b}}(\bar{M}, \bar{N})$ just says that (S1) holds the whole $\psi^1_{\bar{a}|\bar{b}}(\bar{M}, \bar{N})$ also holds.

We now prove the "only if" direction. Suppose $\psi^1_{\bar{a}|\bar{b}}(\bar{M}, \bar{N})$ holds in $\mathcal{N}$. Since $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} = \|\bar{N}^T\|_{\mathrm{per}(\bar{b})} = 0$, we may ignore all the periodic entries in $\bar{a}$ and $\bar{b}$ and assume that $\bar{a}$ and $\bar{b}$ contain only fixed entries, i.e., $\bar{a} = \mathrm{offset}(\bar{a})$ and $\bar{b} = \mathrm{offset}(\bar{b})$.

Our proof is similar to the one of [19, Lemma 7.2] which shows how to construct an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$ for big-enough $\bar{M}|\bar{N}$. For completeness, we repeat the construction here, which we will also see later (e.g., in the proof of Lemma 4.4).

Suppose $\bar{a} \cdot \bar{M} = \bar{b} \cdot \bar{N} = K$. To construct an $\bar{a}|\bar{b}$-biregular graph G with size $\bar{M}|\bar{N}$, we "*partition on one side, merge on the other side, and swap.*" Intuitively, this means that we first construct an $\bar{a}|1$-biregular graph $G = (U, V, E)$ with size $\bar{M}|K$, i.e., the vertices on the left side are partitioned correctly to have degrees $\bar{a}$ and those on the right side all have degree 1. Then, we "merge" vertices on the right side so that they have the correct degrees $\bar{b}$. Since this merging may produce parallel edges between two vertices, we perform "edge swapping" to get rid of them without changing the degree of each vertex.

The details of the construction are as follows. Since $\bar{a} \cdot \bar{M} = K$, it is straightforward to construct an $\bar{a}|1$-biregular graph $G = (U, V, E)$ with size $\bar{M}|K$. Let $\bar{N} = (N_1, \ldots, N_n)$ and $\bar{b} = (b_1, \ldots, b_n)$. To obtain an $\bar{a}|\bar{b}$-biregular graph, we partition $V = V_1 \uplus \cdots \uplus V_n$, where $|V_j| = N_j b_j$ for each $j \in [n]$. This is possible since $K = \bar{b} \cdot \bar{N}$. Then, for each $j \in [n]$, we merge allb_j vertices in V_j into 1 vertex, thus, making its degree b_j. Such merging yields an "almost" $\bar{a}|\bar{b}$-biregular graph, except that it is possible there are parallel edges between two vertices. Here big-enough comes into play, where the condition (a) in Definition 4.2 is applied, i.e., $\max(\|\bar{M}^T\|_{\mathrm{nz}(\bar{a})}, \|\bar{N}^T\|_{\mathrm{nz}(\bar{b})}) \geqslant 2\delta(\bar{a}, \bar{b})^2 + 1$. We will get rid of the parallel edges one by one.

Suppose in-between vertices u and v there are several parallel edges. There are only at most $\delta(\bar{a}, \bar{b})^2$ edges incident to the neighbors of vertex u (including parallel edges). The same holds for neighbors of v. Note that there are at least $\max(\|\bar{M}^T\|_{\mathrm{nz}(\bar{a})}, \|\bar{N}^T\|_{\mathrm{nz}(\bar{b})}) \geqslant 2\delta(\bar{a}, \bar{b})^2 + 1$ edges in G. So there is an edge (w, w') such that both w, w' are not adjacent to either u or v. To get rid of one parallel edge (u, v) between u and v, we replace it and (w, w') by (u, w') and (w, v) (see Figure 2 for an illustration). We perform such edge swapping until there are no parallel edges. Furthermore, such edge swapping does not change the degree of the vertices. ∎

The formula and argument for scenario (S2)*: Creating a "phantom partition" for the period, then merging.* Recall that (S2) states that "there are vertices with periodic degrees on exactly one side." By symmetry, we may assume that the vertices with periodic degrees are on the left. Let the formula $\psi^2_{\bar{a}|\bar{b}}(\bar{x}, \bar{y})$ be defined as

$$\exists z \ \left(\mathrm{offset}(\bar{a}) \cdot \bar{x} + pz = \mathrm{offset}(\bar{b}) \cdot \bar{y}\right) \ \wedge \ \|\bar{x}^T\|_{\mathrm{per}(\bar{a})} \neq 0 \ \wedge \ \|\bar{y}^T\|_{\mathrm{per}(\bar{b})} = 0. \tag{4.2}$$

As in the earlier scenario, the last two conjuncts state that (S2) holds. The first is an edge counting equality, with pz representing the total number of edges added by the periodic components over all elements on the left-hand side.

LEMMA 4.4. *For every pair of degree vectors $\bar{a}$, $\bar{b}$ and for every $\bar{M}|\bar{N}$ big-enough for $\bar{a}|\bar{b}$, the formula $\psi^2_{\bar{a}|\bar{b}}(\bar{M}, \bar{N})$ holds in $\mathcal{N}$ if and only if there is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$ where* (S2) *holds.*

Proof. Let $\bar{a}, \bar{b}$ be degree vectors and $\bar{M}|\bar{N}$ be size vectors big-enough for $\bar{a}|\bar{b}$. We first prove the if direction. Note that if $G = (U, V, E)$ is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$ where (S2) holds, then the number of edges $|E|$ should equal the sum of

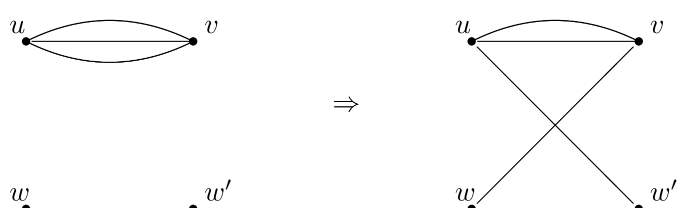

FIG. 2. *Edge swapping used in the proof of Lemma* 4.3. *After swapping there is one less parallel edge between u and v, and the degrees of all vertices stay the same.*

the degrees of the vertices in U, which is $\mathrm{offset}(\bar{a}) \cdot \bar{M} + zp$, for some integer $z \geqslant 0$. Since this quantity must equal the sum of the degrees of the vertices in V, which is $\mathrm{offset}(\bar{b}) \cdot \bar{N}$, we conclude that the first conjunct of $\psi^2_{\bar{a}|\bar{b}}(\bar{M}, \bar{N})$ holds. Since (S2) holds by assumption, the second conjuncts also hold.

We now prove the "only if" direction. Assume that $\psi^2_{\bar{a}|\bar{b}}(\bar{M}, \bar{N})$ holds in $\mathcal{N}$. By (4.2), we have $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} \neq 0$ and $\|\bar{N}^T\|_{\mathrm{per}(\bar{b})} = 0$. Clearly we might as well assume that $\bar{b}$ contains only fixed entries, i.e., $\mathrm{offset}(\bar{b}) = \bar{b}$.

To construct an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$, we "*create a phantom partition for the period, then merge.*" Abusing notation, we denote the value assigned to variable z by z itself. Suppose $\mathrm{offset}(\bar{a}) \cdot \bar{M} + pz = \bar{b} \cdot \bar{N}$. Since $\bar{M}|\bar{N}$ is big-enough for $\bar{a}|\bar{b}$, it follows immediately that $(\bar{M}, z)|\bar{N}$ is big-enough for $(\mathrm{offset}(\bar{a}), p)|\bar{b}$. Applying Lemma 4.3, there is an $(\mathrm{offset}(\bar{a}), p)|\bar{b}$-biregular graph with size $(\bar{M}, z)|\bar{N}$. That is, we have a graph that satisfies our requirements, but there is an additional partition class Z on the left of size z, where the degree of elements is p. Let $G = (U, V, E)$ be such a graph, and let $U = U_0 \uplus U_1 \uplus Z$, where U_0 is the set of vertices whose degrees are from the fixed entries in $\bar{a}$ and U_1 is the set of vertices whose degrees satisfy the periodic entries in $\bar{a}$: in fact, they will initially satisfy these using just the offset. Note that $|U_1| = \|\bar{M}^T\|_{\mathrm{per}(\bar{a})}$ and $|Z| = z$.

We will construct an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$. The idea is to merge the vertices in Z with vertices in U_1. Let $z_0 \in Z$. The number of vertices in U_1 reachable from z_0 in distance 2 is at most $\delta(\bar{a}, \bar{b})^2$. Because $\bar{M}|\bar{N}$ is big-enough for $\bar{a}|\bar{b}$, $|U_1| = \|\bar{M}^T\|_{\mathrm{per}(\bar{a})} \geqslant \delta(\bar{a}, \bar{b})^2 + 1$. Thus, there is a vertex $u \in U_1$ not reachable from z_0 in distance 2; that is, u does not share adjacent vertices with z_0. We merge z_0 and u into one vertex. See Figure 3 for an illustration. Since the degree of z_0 is p, the merging increases the degree of u by p, which does not break our requirement. We perform this merging for each vertex in Z.

Note that the constructed graph G is $\bar{a}|\bar{b}$-biregular, where $\bar{a}$ contains periodic entries and $\bar{b}$ contains only fixed entries. Thus, (S2) holds in G. ∎

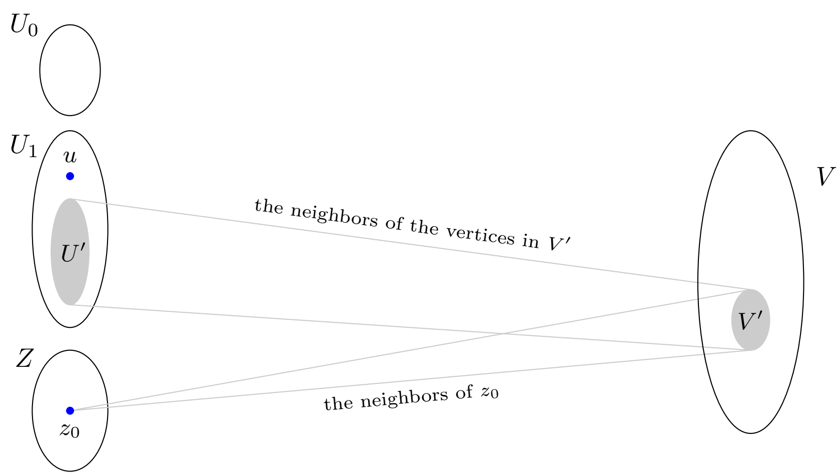

FIG. 3. *Illustration of the choice of the vertices $z_0 \in Z$ and $u \in U_1$. The set V' is the set of the neighbors of z_0. The set U' is the set of the neighbors of the vertices in V' in set U_1, i.e., the set of vertices reachable from z_0 in distance 2. Since $|U_1| \geqslant \delta(\bar{a}, \bar{b})^2 + 1$ and $|U'| \leqslant \delta(\bar{a}, \bar{b})^2$, there is a vertex $u \in U_1 - U'$. We merge z_0 and u into one vertex.*

The formula and argument for scenario (S3)*: Move a multiple of the period entries to one side.* Recall that (S3) states that "there are vertices with (finite) periodic degrees on both sides." Consider the formula $\psi^3_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ that is defined as follows:

(4.3)
$$\exists z_1, z_2\big(\mathrm{offset}(\bar{a})\cdot\bar{x} + pz_1 = \mathrm{offset}(\bar{b})\cdot\bar{y} + pz_2\big) \wedge \|\bar{x}^T\|_{\mathrm{per}(\bar{a})} \neq 0 \wedge \|\bar{y}^T\|_{\mathrm{per}(\bar{b})} \neq 0.$$

LEMMA 4.5. *For every pair of degree vectors $\bar{a},\bar{b}$ and for every $\bar{M}|\bar{N}$ big-enough for $\bar{a}|\bar{b}$, the formula $\psi^3_{\bar{a}|\bar{b}}(\bar{M},\bar{N})$ holds in $\mathcal{N}$ if and only if there is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$ where* (S3) *holds.*

Proof. As before, the if direction is straightforward, so we focus on the only if direction. Suppose $\psi^3_{\bar{a}|\bar{b}}(\bar{M},\bar{N})$ holds in $\mathcal{N}$. If there are witnesses z_1 and z_2 such that $z_1 \geqslant z_2$, we can rewrite the first conjunct as the following:

$$\exists z_1, z_2\big(\mathrm{offset}(\bar{a})\cdot\bar{x} + p(z_1 - z_2) = \mathrm{offset}(\bar{b})\cdot\bar{y}\big).$$

That is, we "*move the multiple of period p to one side,*" i.e., to the left side. Let $\bar{b}'$ denote the vector formed by taking offsets of $\bar{b}$. Thus by definition, $\|\bar{N}^T\|_{\mathrm{per}(\bar{b}')} = 0$. After replacing $z_1 - z_2$ with z, we can apply Lemma 4.4, corresponding to scenario (S2), to $\bar{a}$ and $\bar{b}'$. Applying this tells us that there is an $\bar{a}|\mathrm{offset}(\bar{b})$-biregular graph with size $\bar{M}|\bar{N}$. This graph, of course, is also $\bar{a}|\bar{b}$-biregular.

Note that in this case, i.e., when $z_1 \geqslant z_2$, we are arguing, using the prior characterization and algebra, that when the condition holds we can construct a graph where the degrees on the right-hand side are exactly $\mathrm{offset}(\bar{b})$; that is, we do not need to take advantage of the ability to have a nontrivial period. The proof for the case $z_1 \leqslant z_2$ is analogous. ∎

To wrap up subsection 4.1.2, we define the formula $\psi_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ as follows,

$$\psi^1_{\bar{a}|\bar{b}}(\bar{x},\bar{y}) \vee \psi^2_{\bar{a}|\bar{b}}(\bar{x},\bar{y}) \vee \psi^2_{\bar{b}|\bar{a}}(\bar{y},\bar{x}) \vee \psi^3_{\bar{a}|\bar{b}}(\bar{x},\bar{y}), \tag{4.4}$$

where each formula $\psi^i_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ handles one of the scenarios described above. Combining Lemmas 4.3–4.5, $\psi_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ captures precisely all the big-enough sizes $\bar{M}|\bar{N}$ of an $\bar{a}|\bar{b}$-biregular graph. This is stated formally as Lemma 4.6.

LEMMA 4.6. *For each pair of degree vectors $\bar{a},\bar{b}$ and for each $\bar{M}|\bar{N}$ big-enough for $\bar{a}|\bar{b}$, the formula $\psi_{\bar{a}|\bar{b}}(\bar{M},\bar{N})$ holds in $\mathcal{N}$ if and only if there is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$.*

4.1.3. The formula for the case of not-big-enough sizes: Fixed size encoding. Subsection 4.1.2 gives a formula that captures the existence of 1-color biregular graphs for big-enough sizes. We now turn to sizes that are not-big-enough—that is, when one of the conditions (a)–(c) is violated. When condition (a) is violated, we have restricted the total size of the graph, and thus we can write a formula that simply enumerates all possible valid sizes.

We will first consider the case when (b) is violated, while (a) and (c) hold. If condition (b) is violated, the value of $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})}$ is some r between 1 and $\delta(\bar{a},\bar{b})^2$ and it suffices to show that, for each fixed r between 1 and $\delta(\bar{a},\bar{b})^2$, we can find a formula that works for this r. The idea is that a fixed number of vertices in a graph can be "encoded" as formulas. We will refer to this technique as *fixed size encoding* in the remainder of the paper.

We will define a formula covering the case where each of the following holds:
- $\|\bar{M}^T\|_{\mathrm{nz}(\bar{a})} - \|\bar{M}^T\|_{\mathrm{per}(\bar{a})} \geqslant 2\delta(\bar{a},\bar{b})^2 + 1$.
- $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} = r$ for some fixed r between 1 and $\delta(\bar{a},\bar{b})^2$.
- $\|\bar{N}^T\|_{\mathrm{per}(\bar{b})} = 0$ or $\geqslant \delta(\bar{a},\bar{b})^2 + 1$.

Note that the first bullet item is a slightly stronger requirement than the one required in the definition of big-enough size. However, this does not affect the applicability to the case where (b) is violated and both (a) and (c) hold. If (b) is violated, i.e., $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} = r$, where $1 \leqslant r \leqslant \delta(\bar{a},\bar{b})^2$ and if $\|\bar{M}^T\|_{\mathrm{nz}(\bar{a})} - \|\bar{M}^T\|_{\mathrm{per}(\bar{a})} \leqslant 2\delta(\bar{a},\bar{b})^2$, then $\|\bar{M}^T\|_{\mathrm{nz}(\bar{a})} \leqslant 3\delta(\bar{a},\bar{b})^2$, which means that the number of edges is fixed and all possible sizes of $A|B$-biregular graphs can be simply enumerated.

The formula is defined inductively on r, with the base case $r = 0$. Note that when $r = 0$, $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} = 0$, which means (b) is no longer violated and it falls under the big-enough case. We now give the inductive construction. Let $\bar{a}$ and $\bar{b}$ be degree vectors. For an integer $r \geqslant 0$, define the formula $\phi^r_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ as follows:
- when $r = 0$, let
$$\phi^0_{\bar{a}|\bar{b}}(\bar{x},\bar{y}) := \|\bar{x}^T\|_{\mathrm{per}(\bar{a})} = 0 \ \wedge \ \psi_{\bar{a}|\bar{b}}(\bar{x},\bar{y}),$$
where $\psi_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ is defined in (4.4);
- when $r \geqslant 1$, let
$$\phi^r_{\bar{a}|\bar{b}}(\bar{x},\bar{y}) := \exists s \exists \bar{z}_0 \exists \bar{z}_1 \bigvee_{i \in \mathrm{per}(\bar{a})} \left(\begin{array}{l} x_i \neq 0 \ \wedge \ \bar{z}_0 + \bar{z}_1 = \bar{y} \\ \wedge \ \|\bar{z}_1^T\|_{\mathrm{nz}(\bar{b})} = \mathrm{offset}(a_i) + ps \\ \wedge \ \phi^{r-1}_{\bar{a}|(\bar{b},\bar{b}-\bar{1})}(\bar{x} - \mathbf{e}_i, \bar{z}_0, \bar{z}_1) \end{array} \right), \tag{4.5}$$
where the lengths of $\bar{z}_0$ and $\bar{z}_1$ are the same as $\bar{y}$, $\mathbf{e}_i$ is the unit vector where the ith component is 1, and the subtraction $\bar{b} - \bar{1}$ of degree vectors is the usual elementwise subtraction except the cases $b^{+p} - 1 = (b-1)^{+p}$ for $b > 0$, $0^{+p} - 1 = (p-1)^{+p}$, and $0 - 1 = 0$.

Lemma 4.7. *For every pair of degree vectors $\bar{a},\bar{b}$, for every pair of size vectors $\bar{M},\bar{N}$, and each integer $r \geqslant 0$ such that*
- $\|\bar{M}^T\|_{\mathrm{nz}(\bar{a})} \geqslant 2\delta(\bar{a},\bar{b})^2 + 1 + r$,
- $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} = r$,
- $\|\bar{N}^T\|_{\mathrm{per}(\bar{b})} \geqslant \delta(\bar{a},\bar{b})^2 + 1$,

the formula $\phi^r_{\bar{a}|\bar{b}}(\bar{M},\bar{N})$ holds in $\mathcal{N}$ if and only if there is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$.

Proof. Let $\bar{a},\bar{b}$ be degree vectors and let $\bar{M},\bar{N}$ be size vectors that satisfy the hypothesis. The proof is by induction on r. The base case, as in the formulas, is $r = 0$, and is straightforward by the characterization of big-enough.

Now assume the claim holds inductively for $r - 1 \geqslant 0$, and consider r. We begin with the if direction, which provides the intuition for these formulas. Suppose $G = (U,V,E)$ is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$ that satisfies all the items listed above. Let $U = U_1 \uplus \cdots \uplus U_m$ and $V = V_1 \uplus \cdots \uplus V_n$ be witness partitions.

Since $r \neq 0$ and $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} = r$, there is an $i \in \mathrm{per}(\bar{a})$ such that $U_i \neq \emptyset$. Choose $u \in U_i$. Based on this u, for each $j \in [n]$ we define Z_j to be the set of vertices in V_j adjacent to u. Figure 4 illustrates the situation.

If we omit the vertex u and all its adjacent edges, then, for each $j \in [n]$, every vertex in Z_j has degree $b_j - 1$. Note here that, for each j where $Z_j \neq \emptyset$, $b_j > 0$ since u is adjacent to the vertices in $Z_j \subseteq V_j$. Thus, we are left with an $\bar{a}|(\bar{b},\bar{b}-\bar{1})$-biregular graph with size $(\bar{M} - \mathbf{e}_i)|(\bar{K}_0,\bar{K}_1)$, where $\bar{K}_0 = (|V_1| - |Z_1|,\ldots,|V_n| - |Z_n|)$ and $\bar{K}_1 = (|Z_1|,\ldots,|Z_n|)$. Also note that

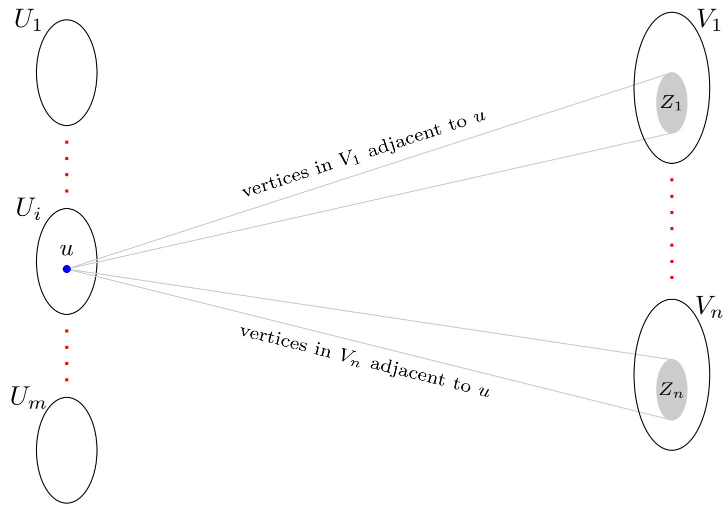

FIG. 4. *Illustration of why the formula for the not-big-enoughcase is a necessary condition. Note: color appears only in the online article.*

- $\|(\bar{M}-\mathbf{e}_i)^T\|_{\mathrm{nz}(\bar{a})} = \|\bar{M}^T\|_{\mathrm{nz}(\bar{a})} - 1 \geqslant 2\delta(\bar{a},\bar{b})^2 + 1 + (r-1)$.
  The equality comes from the fact that $i \in \mathrm{per}(\bar{a})$, hence $i \in \mathrm{nz}(\bar{a})$ and $\|\mathbf{e}_i^T\|_{\mathrm{nz}(\bar{a})} = 1$, which implies the equality. The inequality comes from the assumption that $\|\bar{M}^T\|_{\mathrm{nz}(\bar{a})} \geqslant 2\delta(\bar{a},\bar{b})^2 + 1 + r$;
- $\|(\bar{M}-\mathbf{e}_i)^T\|_{\mathrm{per}(\bar{a})} = r-1$.
  This comes from the assumptions that $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} = r$ and $i \in \mathrm{per}(\bar{a})$;
- $\|(\bar{K}_0,\bar{K}_1)^T\|_{\mathrm{per}(\bar{b},\bar{b}-\bar{1})} = \|\bar{N}^T\|_{\mathrm{per}(\bar{b})} \geqslant \delta(\bar{a},\bar{b})^2 + 1$.
  The equality comes from the fact that $\bar{N} = \bar{K}_0 + \bar{K}_1$, while periodic entries in $\bar{b}$ stay periodic in $\bar{b}-\bar{1}$. The inequality is the assumption that $\|\bar{N}^T\|_{\mathrm{per}(\bar{b})} \geqslant \delta(\bar{a},\bar{b})^2 + 1$.

The items above tell us that $(\bar{M}-\mathbf{e}_i)|(\bar{K}_0,\bar{K}_1)$ satisfies the hypothesis of the lemma (w.r.t. to the degree vectors $\bar{a}|(\bar{b},\bar{b}-\bar{1})$). Thus, we can apply the induction hypothesis and obtain that $\phi^{r-1}_{\bar{a}|(\bar{b},\bar{b}-\bar{1})}(\bar{M}-\mathbf{e}_i,\bar{K}_0,\bar{K}_1)$ holds. Moreover, since $i \in \mathrm{per}(\bar{a})$, the degree of u is $\mathrm{offset}(a_i) + ps$ for some s and hence $\|\bar{K}_1^T\| = \mathrm{offset}(a_i) + ps$. Since, by construction, every vertex in each Z_j is adjacent to u, $|Z_j| = 0$ whenever $b_j = 0$, so, $\|\bar{K}_1^T\| = \mathrm{offset}(a_i) + ps$ implies that $\|\bar{K}_1^T\|_{\mathrm{nz}(\bar{b})} = \mathrm{offset}(a_i) + ps$, and therefore $\phi^r_{\bar{a}|\bar{b}}(\bar{M},\bar{N})$ holds, with the witnessing $\bar{z}_0$ and $\bar{z}_1$ being $\bar{K}_0$ and $\bar{K}_1$, respectively.

For the only if direction, suppose $\phi^r_{\bar{a}|\bar{b}}(\bar{M},\bar{N})$ holds. Then we can fix some $s, \bar{z}_0, \bar{z}_1$, and $i \in \mathrm{per}(\bar{a})$ such that

(a) $M_i \neq 0$;
(b) $\mathrm{offset}(a_i) + ps = \|\bar{z}_1^T\|_{\mathrm{nz}(\bar{b})}$;
(c) $\bar{z}_0 + \bar{z}_1 = \bar{N}$;
(d) $\phi^{r-1}_{\bar{a}|(\bar{b},\bar{b}-\bar{1})}(\bar{M}-\mathbf{e}_i,\bar{z}_0,\bar{z}_1)$ holds.

We prove from this that a biregular graph of the appropriate size exists.

By similar reasoning as in the previous case (i.e., the if case), the following hold:

- $\|(\bar{M}-\mathbf{e}_i)^T\|_{\mathrm{nz}(\bar{a})} = \|\bar{M}^T\|_{\mathrm{nz}(\bar{a})} - 1 \geqslant 2\delta(\bar{a},\bar{b})^2 + 1 + (r-1)$;
- $\|(\bar{M}-\mathbf{e}_i)^T\|_{\mathrm{per}(\bar{a})} = r-1$;
- $\|(\bar{z}_0,\bar{z}_1)^T\|_{\mathrm{per}(\bar{b},\bar{b}-\bar{1})} = \|\bar{N}^T\|_{\mathrm{per}(\bar{b})} \geqslant \delta(\bar{a},\bar{b})^2 + 1$.

That is, $(\bar{M}-\mathbf{e}_i)|(\bar{z}_0,\bar{z}_1)$ satisfies the hypothesis of the lemma (w.r.t. to the degree vectors $\bar{a}|(\bar{b},\bar{b}-\bar{1})$). Thus, we can apply the induction hypothesis and obtain an $\bar{a}|(\bar{b},\bar{b}-\bar{1})$-biregular graph $G=(U,V,E)$ with size $(\bar{M}-\mathbf{e}_i)|(\bar{z}_0,\bar{z}_1)$. Let $U=U_1 \uplus \cdots \uplus U_m$ and $V=V_{0,1}\uplus\cdots\uplus V_{0,n}\uplus V_{1,1}\uplus\cdots\uplus V_{1,n}$ be the witness partitions. Note that the degrees of the vertices in $V_{1,1}\uplus\cdots\uplus V_{1,n}$ are $\bar{b}-\bar{1}$.

Let u be a fresh vertex. We construct an $\bar{a}|\bar{b}$-biregular graph $G'=(U\cup\{u\},V,E')$, by connecting u with every vertex in $\bigcup_{j\in\mathrm{nz}(\bar{b})} V_{1,j}$. This makes the degree of the vertices in $V_{1,1}\uplus\cdots\uplus V_{1,n}$ become $\bar{b}$. The formula states that $\|\bar{z}_1^T\|_{\mathrm{nz}(\bar{b})}=\mathrm{offset}(a_i)+ps$; thus, the degree of u is $\mathrm{offset}(a_i)+ps$, which satisfies the requirement for a vertex to be in U_i. Moreover, $\bar{z}_0+\bar{z}_1=\bar{N}$. Thus, the graph G' has size $\bar{M}|\bar{N}$. As witness partition for G' we use the U_j on the left, while on the right each $V_{0,j}\cup V_{1,j}$ becomes a single partition element. ∎

The case where (a) and (b) hold, but (c) is violated is handled symmetrically.

Next, we consider the case when (a) holds, but both (b) and (c) are violated. The treatment is similar to the previous case. We will define a formula for the case where all of the following hold.

- $\|\bar{M}^T\|_{\mathrm{nz}(\bar{a})}-\|\bar{M}^T\|_{\mathrm{per}(\bar{a})}\geqslant 2\delta(\bar{a},\bar{b})^2+1$.
- $\|\bar{N}^T\|_{\mathrm{nz}(\bar{b})}-\|\bar{N}^T\|_{\mathrm{per}(\bar{b})}\geqslant 2\delta(\bar{a},\bar{b})^2+1$.
- $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})}=r_1$ for some fixed r_1 between 0 and $\delta(\bar{a},\bar{b})^2$.
- $\|\bar{N}^T\|_{\mathrm{per}(\bar{b})}=r_2$ for some fixed r_2 between 0 and $\delta(\bar{a},\bar{b})^2$.

The formula is defined inductively on r_2 with the base case $r_2=0$. Note that when $r_2=0$, $\|\bar{N}^T\|_{\mathrm{per}(\bar{b})}=0$, which means (c) is no longer violated and it falls under the previous case. Define the formula $\phi^{r_1,r_2}_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ as follows:

- when $r_2=0$, let
$$\phi^{r_1,0}_{\bar{a}|\bar{b}}(\bar{x},\bar{y}) := \ \|\bar{y}^T\|_{\mathrm{per}(\bar{b})}=0 \ \wedge \ \phi^{r_1}_{\bar{a}|\bar{b}}(\bar{x},\bar{y}),$$
where $\phi^{r_1}_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ is defined in the previous case;
- when $r_2\geqslant 1$, let
$$(4.6)\quad \phi^{r_1,r_2}_{\bar{a}|\bar{b}}(\bar{x},\bar{y}) \ := \ \exists s\exists\bar{z}_0\exists\bar{z}_1 \bigvee_{i\in\mathrm{per}(\bar{b})} \left(\begin{array}{l} y_i\neq 0 \ \wedge \ \bar{z}_0+\bar{z}_1=\bar{x} \\ \wedge\ \|\bar{z}_1^T\|_{\mathrm{nz}(\bar{a})}=\mathrm{offset}(b_i)+ps \\ \wedge\ \phi^{r_1,r_2-1}_{(\bar{a},\bar{a}-\bar{1})|\bar{b}}(\bar{z}_0,\bar{z}_1,\bar{y}-\mathbf{e}_i)\end{array}\right).$$
Here the lengths of $\bar{z}_0$ and $\bar{z}_1$ are the same as $\bar{x}$, $\mathbf{e}_i$ is the unit vector where the ith component is 1, and the subtraction $\bar{a}-\bar{1}$ of degree vectors is the same as in the earlier case.

Note that the formula $\phi^{r_1,r_2}_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ is defined as in the previous case, but the roles of $\bar{a},\bar{x}$ and $\bar{b},\bar{y}$ are reversed and the base case is now the formula $\phi^{r_1,0}(\bar{x},\bar{y})$.

Lemma 4.8. *For every pair of degree vectors $\bar{a},\bar{b}$, for every pair of size vectors $\bar{M},\bar{N}$, and each integer $r_1,r_2\geqslant 0$ such that*

- $\|\bar{M}^T\|_{\mathrm{nz}(\bar{a})}\geqslant 2\delta(\bar{a},\bar{b})^2+1+r_1$,
- $\|\bar{N}^T\|_{\mathrm{nz}(\bar{b})}\geqslant 2\delta(\bar{a},\bar{b})^2+1+r_2$,
- $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})}=r_1$,
- $\|\bar{N}^T\|_{\mathrm{per}(\bar{b})}=r_2$,

the formula $\phi^{r_1,r_2}_{\bar{a}|\bar{b}}(\bar{M},\bar{N})$ holds in $\mathcal{N}$ if and only if there is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$.

The proof of Lemma 4.8 is similar to Lemma 4.7, hence is omitted.

We have completed the case of fixed r_1, r_2. As mentioned above, this suffices to give the entire not-big-enough case, via enumerating solutions for each of the finitely many possible values of r_1, r_2.

To wrap up this section, we define the formula $\mathsf{bireg}_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ required in Lemma 4.1 to characterize solutions in the 1-color case without the completeness requirements

$$\mathsf{bireg}_{\bar{a}|\bar{b}}(\bar{x},\bar{y}) := \psi_{\bar{a}|\bar{b}}(\bar{x},\bar{y}) \ \vee\ \phi_{\bar{a}|\bar{b}}(\bar{x},\bar{y}) \ \vee \bigvee_{r=1}^{\delta(\bar{a},\bar{b})^2} \left(\phi^r_{\bar{a}|\bar{b}}(\bar{x},\bar{y}) \vee \phi^r_{\bar{b}|\bar{a}}(\bar{y},\bar{x})\right)$$
$$\vee \bigvee_{0\leqslant r_1,r_2\leqslant\delta(\bar{a},\bar{b})^2} \phi^{r_1,r_2}_{\bar{a}|\bar{b}}(\bar{x},\bar{y}),$$

where $\psi_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ is defined in (4.4) to deal with the big-enough sizes, $\phi_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ is the formula enumerating all valid sizes when condition (a) is violated, the formulas in the second last disjunction deal with the not-big-enough cases when exactly one of the conditions (b) or (c) is violated as defined in (4.5), and the formulas in the final disjunctions deal with the other not-big-enough cases as defined in (4.6). The correctness of the construction follows immediately from Lemmas 4.6 and 4.8.

4.2. The proof in the 1-color case for biregular digraphs. For the regular digraph case, we can essentially use the same argument as in the biregular case. Recall that we define digraphs as without any self-loop. Thus, a digraph can be viewed as a bipartite graph by splitting every vertex u into two vertices, where one is adjacent to all the incoming edges, and the other to all the outgoing edges; see Figure 5. Conversely, a bipartite graph can be viewed as a digraph by merging every two vertices into one vertex. Thus, $\bar{a}|\bar{b}$-regular digraphs with size $\bar{M}$ can be characterized as $\bar{a}|\bar{b}$-biregular graphs with size $\bar{M}|\bar{M}$ (see [19, section 8] for a similar construction when the degrees are fixed).

To illustrate, we will give a formula that captures the sizes of $a^{+p}|b$-regular digraph. Consider the formula

$$\varphi(x) := \exists z\ ax + pz = bx.$$

We claim that for every $M \geqslant bp + p + 1$, the formula $\varphi(M)$ holds in $\mathcal{N}_\infty$ if and only if there is $a^{+p}|b$-regular digraph of size M.

For the only if direction, suppose there is $a^{+p}|b$-regular digraph G of size M. Splitting each vertex in G into 2—as illustrated in Figure 5—we obtain $a^{+p}|b$-biregular graph of size $M|M$, which by Lemma 4.4, implies that $\varphi(M)$ holds in $\mathcal{N}_\infty$.

For the if direction, suppose $\varphi(M)$ holds in $\mathcal{N}_\infty$. By Lemma 4.3, there is an $(a,p)|b$-biregular graph $G = (U, V, E)$ of size $(M, z)|M$. Let $U = U_0 \uplus U_1$, where U_0 is the set of vertices of degree a and U_1 is the set of vertices of degree p, the phantom partition to be merged with U_0 (using the same merging technique as in Lemma 4.4). Let $u_1, u_2, \ldots$ and $v_1, v_2, \ldots$ be the enumeration of vertices in U_0 and V, respectively.

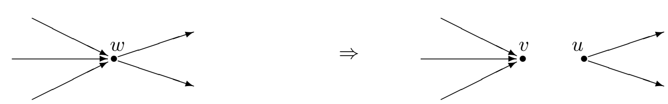

Fig. 5. *Splitting a vertex w in a digraph G into two vertices u and v in G'. One is adjacent to all the outgoing edges and the other to all the incoming edges.*

We will call each v_i the mirror image of u_i and, similarly, each u_i is the mirror image of v_i for every $i \geqslant 1$.

To obtain $a^{+p}|b$-regular digraph of size M, we perform the following steps:

1. Ensure that each u_i is not adjacent to its mirror image in G.
   This can be achieved in the same manner by the *edge swapping* technique in Lemma 4.3. (See also Figure 2.)
2. Merge the vertices in the phantom partition U_1 with vertices in U_0 as follows. For each vertex $w \in U_1$, we merge it with a vertex $u \in U_0$ where u is not reachable from w in distance 2 and u is not the mirror image of the vertices adjacent to w. Such a u exists since $M \geqslant bp + p + 1$. Note that since u is not the mirror image of the vertices adjacent to w, after the merging of w and u, each vertex in U_0 is still not adjacent to its mirror image in V.
   Thus, we obtain $a^{+p}|b$-biregular graph $G' = (U_0, V, E')$ of size $M|M$.
3. Orient all the edges in E' from left to right and merge each vertex in U_0 with its mirror image in V, thus, obtaining an $a^{+p}|b$-regular digraph of size M.
   Note that since each vertex in U_0 is not adjacent to its mirror image in G', there is noself-loop in the digraph.

Some remarks on the general case versus the 1-color case. To conclude this section, we stress that although the 1-color case contains many of the key ideas, the multicolor case requires a finer analysis to deal with the big-enough case, and also may benefit from a reduction that allows one to restrict the analysis to matrices of a very special form that we call "simple matrices". Note that our definition of a multicolor graph requires the edges of different colors to be disjoint, which imposes additional correlations between sizes on top of those one would get from considering each color in isolation. We will present these details in the following sections.

5. Proof of Theorem 3.2 for the case of "simple" matrices and without the completeness requirement.

This section will provide the construction of the Presburger formula for the case where the matrices A and B may have multiple colors, but are what we call *simple* matrices, defined formally in Definition 5.1, and where the requirement of being complete is dropped. Here it is useful to recall that a fixed entry is of the form $a \in \mathbb{N}$ and a periodic entry is of the form a^{+p}.

DEFINITION 5.1. *A degree matrix A is simple if every row consists of either only periodic entries or only fixed entries.*

That is, for every fixed edge color, either each partition is constrained using fixed degree constraints on each vertex, or each partition is only "loosely constrained" with a periodic constraint on each vertex. We devote this section to the proof of the following lemma, which only deals with finite graphs. The extension to general graphs can be found in Appendix C.

LEMMA 5.2. *For every pair of simple matrices $A \in \mathbb{N}_{+p}^{t\times m}$ and $B \in \mathbb{N}_{+p}^{t\times n}$, there exists an (effectively computable) existential Presburger formula $\mathsf{bireg}_{A|B}(\bar{x}, \bar{y})$ such that, for every pair of size vectors $\bar{M} \in \mathbb{N}^m$ and $\bar{N} \in \mathbb{N}^n$, the formula $\mathsf{bireg}_{A|B}(\bar{M}, \bar{N})$ holds in $\mathcal{N}$ if and only if there is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$.*

This section is organized as follows. We introduce the proper notation in subsection 5.1. In the setting with multiple colors we also need to introduce big-enough sizes and "extra-big-enough" sizes. The big-enough sizes are defined only for the matrices $A|B$ whose entries are all fixed, whereas the extra-big-enough sizes are defined for the matrices $A|B$ whose entries can be fixed and periodic. As in the 1-color case, the formula $\mathsf{bireg}_{A|B}(\bar{x}, \bar{y})$ is divided into three cases:

(1) For big-enough sizes and when the degree matrices contain only fixed entries, dealt with in section 5.2.
(2) For extra-big-enough sizes and when the degree matrices may contain fixed and periodic entries, in section 5.3.
(3) For not-big-enough/not-extra-big-enough sizes, in section 5.4.

Lemma 5.2 can then be proven by combining these cases, as we show in section 5.5.

Briefly, the formula for case (1) is the same as the one in [19, Theorem 7.4]. However, the proof we give here is more straightforward. The formula for case (2) is a generalization of the formula for case (1) and we will use techniques such as "*creating a phantom partition for the period, then merging*" (Lemma 4.4); "*move a multiple of the period entries to one side*" (Lemma 4.5), and *edge swapping* (Lemma 4.3). As with the 1-color case, the purpose of extra-big-enough sizes is to enable us to perform these techniques without violating the requirement of $A|B$-biregularity. Finally, the formula for not-extra-big-enough sizes is a straightforward generalization of the "*fixed size encoding*" presented in subsection 4.1.3.

5.1. Notation and terminology. As before, the term vectors means row vectors and we use $\bar{x}, \bar{y}, \bar{z}$ (possibly indexed) to denote vectors of variables, and $\bar{M}, \bar{N}$ to denote size vectors.

Since we are now transitioning to general multicolor graphs, we will use matrix notation, where matrices are primarily used to describe the degrees of vertices. We will often call the matrices *degree matrices.* We use $\cdot$ to denote matrix multiplication. When we perform matrix multiplication, we always assume that the sizes of the operands are appropriate. We write I_t to denote the identity matrix with size $t \times t$.

The transpose of a matrix A is denoted by A^T. The entry in row i and column j is $A_{i,j}$. We write $A_{i,*}$ and $A_{*,j}$ to denote the ith row and jth column of A, respectively. The numbering of the rows and columns of a matrix starts from 1.

As before, we call an entry $A_{i,j}$ a *fixed* entry, if it is some $a \in \mathbb{N}$. Otherwise, it is called a *periodic* entry, i.e., an entry of the form a^{+p}. The *offset* of A, denoted by offset(A), is the matrix obtained by replacing every entry $A_{i,j}$ with offset$(A_{i,j})$. Of course, if A does not contain any periodic entry, then offset(A) is A itself.

For a matrix A (with t rows and m columns) that contains only fixed entries, its norm is defined as $\|A\| = \max_{j\in[m]} \sum_{i=1}^{t} A_{i,j}$. This is the standard 1-norm. Of course, a vector $\bar{a}$ (of fixed entries) can be viewed as a 1 row matrix. Thus, for $\bar{a} = (a_1, \ldots, a_m)$, its norm is $\|\bar{a}\| = \max(a_1, \ldots, a_m)$ and the norm of its transpose is $\|\bar{a}^T\| = \sum_{i=1}^{m} a_i$. For matrices A and B that contain only fixed entries, $\delta(A, B)$ denotes $\max(\|A\|, \|B\|)$. If they contain periodic entries, $\delta(A, B)$ denotes $\max(\|\text{offset}(A)\|, \|\text{offset}(B)\|, p)$. Note that $\delta(A, B)$ is actually the generalization of the $\delta(\bar{a}, \bar{b})$ introduced in section 4.1.1 for the 1-color case.

If A and B are matrices with the same number of columns, $\left(\begin{smallmatrix} A \\ B \end{smallmatrix}\right)$ denotes the matrix where the first sequence of rows is A and the next sequence of rows is B. Likewise, if A and B have the same number of rows, (A, B) is the matrix where the first sequence of columns is A and the next sequence of columns is B.

For degree matrices A and B (with entries from $\mathbb{N}_{+p}$ and the same number of rows), and for all size vectors $\bar{M}$ and $\bar{N}$, we say that $\bar{M}|\bar{N}$ *is appropriate for* $A|B$, if the length of $\bar{M}$ is the same as the number of columns of A and the length of $\bar{N}$ is the same as the number of columns of B. Since we will only use degree matrices A and B to describe $A|B$-biregular graphs (or $A|B$-regular digraphs), in the rest of the paper, whenever we use the notation $A|B$, we implicitly assume that A and B have the same number of rows. Moreover, unless indicated otherwise, entries in degree matrices always come from $\mathbb{N}_{+p}$.

Next, we generalize the notion of big-enough in section 4. The distinction between big-enough and not-big-enough size vectors used for the 1-color case in section 4 will need to be refined.

DEFINITION 5.3. *Let A and B be degree matrices with t rows whose entries are all fixed entries, i.e., from $\mathbb{N}$. For size vectors $\bar{M}$ and $\bar{N}$, where $\bar{M}|\bar{N}$ is appropriate for $A|B$, $\bar{M}|\bar{N}$ is* big-enough *for $A|B$ if the following holds for every $i \in [t]$:*

(a) $\max(\|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})}, \|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})}) \geqslant 2\delta(A,B)^2 + 1$.

DEFINITION 5.4. *Let A and B be simple degree matrices with t rows. Let $\bar{M}$ and $\bar{N}$ be size vectors where $\bar{M}|\bar{N}$ is appropriate for $A|B$. We say that $\bar{M}|\bar{N}$* is extra-big-enough for *$A|B$, if each of the following holds, for every $i \in [t]$:*

(a) $\max(\|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})}, \|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})}) \geqslant 8t^2\delta(A,B)^4 + 1$;
(b) $\|\bar{M}^T\|_{\mathrm{per}(A_{i,*})} = 0$ *or* $\|\bar{M}^T\|_{\mathrm{per}(A_{i,*})} \geqslant \delta(A,B)^2 + 1$;
(c) $\|\bar{N}^T\|_{\mathrm{per}(B_{i,*})} = 0$ *or* $\|\bar{N}^T\|_{\mathrm{per}(B_{i,*})} \geqslant \delta(A,B)^2 + 1$.

Note that since A is a simple matrix, for each color $i \in [t]$, either $\mathrm{per}(A_{i,*}) = \emptyset$ or $\mathrm{per}(A_{i,*}) = [m]$, where m is the number of columns in A. The first case is equivalent to $\|\bar{M}^T\|_{\mathrm{per}(A_{i,*})} = \|\bar{M}^T\| = 0$ in condition (b), while the second case is equivalent to $\|\bar{M}^T\|_{\mathrm{per}(A_{i,*})} = \|\bar{M}^T\| \geqslant \delta(A,B)^2 + 1$. The same property also holds for matrix B and size vector $\bar{N}$. Thus, conditions (a)–(c) can be equivalently restated as

- $\max(\|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})}, \|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})}) \geqslant 8t^2\delta(A,B)^4 + 1$ for every $i \in [t]$;
- if A contains periodic entries, then $\|\bar{M}^T\| \geqslant \delta(A,B)^2 + 1$;
- if B contains periodic entries, then $\|\bar{N}^T\| \geqslant \delta(A,B)^2 + 1$.

This is the version we will use in arguments below. The formulation in Definition 5.4 was presented only to highlight the generalization from the 1-color case in Definition 4.2.

When we say $\bar{M}|\bar{N}$ is big/extra-big-enough for $A|B$, we implicitly assume that $\bar{M}|\bar{N}$ is appropriate for $A|B$.

Remark 5.5. Some basic observations:

- The notion of big-enough is defined just on matrices $A|B$ which contain only fixed entries.
- Definition 5.3 is a direct generalization of Definition 4.2 for the case without periodic degrees, where $\bar{M}|\bar{N}$ is big-enough for $A|B$, if $\bar{M}|\bar{N}$ is big-enough for every color, i.e., $\bar{M}|\bar{N}$ is big-enough for degree vector $A_{i,*}|B_{i,*}$ (in the sense of Definition 4.2) for every row i.
- In the notion of extra-big-enough, in Definition 5.4 condition (a) requires that $\max(\|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})}, \|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})})$ is at least $8t^2\delta(A,B)^4 + 1$, which is quartic in $\delta(A,B)$, a jump from quadratic for the 1-color case. The reason is purely technical, because in multiple color graphs, in some cases periodic entries can be reduced to fixed entries but with quadratic blowup on the matrix entries.
- Of course, extra-big-enough is stronger than big-enough.

Informally, big-enough entries are those that will allow the analogous results to Lemma 4.3 from the 1-color case, which concerned fixed-degree constraints, to go through. Extra big-enough will have some additional margin over big-enough, which will allow us to handle the case of matrices with periodic entries by reduction to the fixed-entry case.

5.2. Proof of Lemma 5.2 for big-enough sizes, when the degree matrices are simple matrices containing only fixed entries. Let A and B be degree matrices with t rows that contain only fixed entries. Note that in this case, A and B

are also simple matrices. In fact, they are just a special case of simple matrices that do not contain periodic entries, and the corresponding big-enough sizes are defined in Definition 5.3. Consider the formula:

$$\Psi^1_{A|B}(\bar{x},\bar{y}) := A \cdot \bar{x}^T = B \cdot \bar{y}^T. \tag{5.1}$$

This formula is a generalization of (4.1) to the case of multiple color graphs for $\bar{a}$ and $\bar{b}$ without periodic entries.

LEMMA 5.6. *For every pair of degree matrices A, B that contain only fixed entries and for every pair of size vectors $\bar{M}, \bar{N}$ such that $\bar{M}|\bar{N}$ is big-enough for $A|B$, the formula $\Psi^1_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}$ if and only if there is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$.*

Proof. Let A and B be degree matrices with t rows, containing only fixed entries. Let $\bar{M}|\bar{N}$ be big-enough for $A|B$.

We argue for the if direction. Let $G = (U, V, E_1, \ldots, E_t)$ be an $A|B$-biregular graph with size $\bar{M}|\bar{N}$. The equality, as in the analogous 1-color case, comes from the "edge counting equality," i.e., both $A \cdot \bar{M}^T$ and $B \cdot \bar{N}^T$ simply "count" the number of edges in each color, i.e., $A \cdot \bar{M}^T = (|E_1|, \ldots, |E_t|)^T = B \cdot \bar{N}^T$. Thus, $\Psi^1_{A|B}(\bar{M},\bar{N})$ holds.

We now show the only if direction. Suppose $\Psi^1_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}$, i.e., $A \cdot \bar{M}^T = B \cdot \bar{N}^T$. We will show that there is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$.

The proof is by induction on t. The base case $t = 1$ has been shown in Lemma 4.3. For the induction hypothesis, we assume the lemma holds when the number of colors is less than t.

Let A' and B' be the degree matrices obtained by omitting the last row in A and B, respectively. Since $\bar{M}|\bar{N}$ is big-enough for $A|B$, we infer that $\bar{M}|\bar{N}$ is big-enough for $A'|B'$. Applying the induction hypothesis, there is an $A'|B'$-biregular graph $G' = (U', V', E_1, \ldots, E_{t-1})$ with size $\bar{M}|\bar{N}$.

Similarly, since $\bar{M}|\bar{N}$ is big-enough for $A|B$, it is big-enough for $A_{t,*}|B_{t,*}$ (in the sense of Definition 4.2). Recall that $A_{t,*}$ and $B_{t,*}$ are the last rows of A and B. Applying the induction hypothesis, there is an $A_{t,*}|B_{t,*}$-biregular graph $G'' = (U'', V'', E_t)$ with size $\bar{M}|\bar{N}$. Since G' and G'' have the same size, we can assume that $U'' = U'$ and $V'' = V'$.

To obtain the desired $A|B$-biregular graph, we first merge the two graphs, obtaining a single graph $G = (U, V, E_1, \ldots, E_t)$. Such a graph G is almost $A|B$-biregular, except that it is possible we have an edge (u, v) which is in $E_1 \cup \cdots \cup E_{t-1}$ as well as in E_t. Here we will make use of the edge swapping technique adapted from Lemma 4.3.

Recall that $\delta(A, B) = \max(\|A\|, \|B\|)$. Thus, there are only at most $\delta(A, B)^2$ edges incident to the neighbors (via any of $E_1, \ldots, E_t$-edges) of vertex u. The same holds for neighbors of v. Since $\bar{M}|\bar{N}$ is big-enough for $A|B$, there are at least $\max(\|\bar{M}^T\|_{\mathrm{nz}(A_{t,*})}, \|\bar{N}^T\|_{\mathrm{nz}(B_{t,*})}) \geqslant 2\delta(A, B)^2 + 1$ E_t-edges in G. So there is an E_t-edge (w, w') such that both w, w' are not adjacent (via any of $E_1, \ldots, E_t$-edges) to either u or v. We can perform edge swapping where we omit the edges $(u, v), (w, w')$ from E_t, but add $(u, w'), (w, v)$ into E_t. This edge swapping does not effect the degree of any of the vertices u, v, w, w'. ∎

5.3. Proof of Lemma 5.2 for extra-big-enough sizes. In this section we will present the construction of the formula for Lemma 5.2 that captures all the extra-big-enough sizes. For illustration, we start with subsection 5.3.1 where we consider a special case when the degree matrices A and B contain only 1 column and 2 rows, the proof of which already contains all the essential ideas required for the proof this case. Then, in subsection 5.3.2, we present the general formula for the extra-big-enough sizes for Lemma 5.2.

5.3.1. A special case to illustrate the main ideas. We consider the two-color case, i.e., $t = 2$, and the degree matrices $A_0 = \binom{a_1}{a_2^{+p}}$ and $B_0 = \binom{b_1^{+p}}{b_2}$, where a_1, a_2, b_1, b_2 are all nonzero. Both A_0 and B_0 have only one column—that is, only a single partition, whose size will be the size of one side of the bipartite graph. So it is trivial that every row contains either only fixed entries or only periodic entries. Hence both are simple matrices.

We will now present the formula $\psi_0(x,y)$ that captures all possible extra-big-enough sizes $M|N$ of $A_0|B_0$-biregular graphs:

$$\psi_0(x,y) := \exists z_1 \exists z_2 \; a_1 x = b_1 y + p z_1 \;\wedge\; a_2 x + p z_2 = b_2 y.$$

Equivalently, we can write $\psi_0(x,y)$ in matrix form,

$$\psi_0(x,y) := \exists z_1 \exists z_2 \quad C \begin{pmatrix} x \\ z_2 \end{pmatrix} \;=\; D \begin{pmatrix} y \\ z_1 \end{pmatrix},$$

where $C = \left(\begin{smallmatrix} a_1 & 0 \\ a_2 & p \end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix} b_1 & p \\ b_2 & 0 \end{smallmatrix}\right)$. Note that C and D contain only fixed entries. The following lemma will be useful.

LEMMA 5.7. *For every pair of integers $M, N \geqslant 0$, if $M|N$ is extra-big-enough for $A_0|B_0$, then, for all integers $K, L \geqslant 0$, $(M,K)|(N,L)$ is big-enough for $C|D$.*

Proof. The proof is straightforward from the definitions of extra-big-enough, big-enough, $\delta(A_0, B_0)$, and $\delta(C, D)$. ∎

We now show that $\psi_0(x,y)$ captures all possible extra-big-enough sizes $M|N$ of $A_0|B_0$-biregular graphs, stated formally in Lemma 5.8. The proof actually contains all the essential ideas required for the proof of Lemma 5.2.

LEMMA 5.8. *For every pair of integers $M, N \geqslant 0$, if $M|N$ is extra-big-enough for $A_0|B_0$, then the formula $\psi_0(M,N)$ holds in $\mathcal{N}$ if and only if there is an $A_0|B_0$-biregular graph with size $M|N$.*

Proof. Let $M|N$ be extra-big-enough for $A_0|B_0$. Again, the if direction follows immediately from the edge counting equality. So, we focus on the only if direction. Suppose $\psi_0(M,N)$ holds, i.e., there are $K, L \geqslant 0$ such that

$$a_1 M \;=\; b_1 N + pK, \tag{5.2}$$

$$a_2 M + pL \;=\; b_2 N. \tag{5.3}$$

Since $M|N$ is extra-big-enough for $A_0|B_0$, by Lemma 5.7, $(M,L)|(N,K)$ is big-enough for $C|D$. By Lemma 5.6, there is a $C|D$-biregular graph $G = (U, V, E_1, E_2)$ of size $(M,L)|(N,K)$. Let $U = U_0 \uplus U_1$ and $V = V_0 \uplus V_1$ be the witness partitions, where $(|U_0|, |U_1|) = (M, L)$ and $(|V_0|, |V_1|) = (N, K)$ and the degree of every vertex is as follows:

- every vertex in U_0 has E_1-degree a_1 and E_2-degree a_2;
- every vertex in U_1 has E_1-degree 0 and E_2-degree p;
- every vertex in V_0 has E_1-degree b_1 and E_2-degree b_2;
- every vertex in V_1 has E_1-degree p and E_2-degree 0.

We will show how to merge every vertex in the "phantom" partition U_1 with some vertex in U_0 and likewise, merge every vertex in the phantom partition V_1 with some vertex in V_0.

We consider two cases: (a) at least one of K or L is zero; (b) both K and L are not zero.

Case (a)*: When at least one of K or L is zero.* We may assume that $K = 0$, i.e., $V_1 = \emptyset$. Hence we may consider $G = (U, V, E_1, E_2)$ as a $C|\text{offset}(B_0)$-biregular graph of size $(M, L)|N$. We will use the same merging technique as in scenario (S2) in subsection 4.1.2.

Let $w \in U_1$. The number of vertices in U_0 reachable from w in distance 2 is at most $\delta(A_0, B_0)^2$. Due to the condition that $(M, L)|N$ is big-enough for $C|\text{offset}(B_0)$, we have $|U_0| = M \geqslant \delta(A_0, B_0)^2 + 1$. Thus, there is a vertex $u \in U_0$ not reachable from w in distance 2: that is, u does not share adjacent vertices with w. We merge z_0 and u into one vertex. Since the E_1-degree of w is 0 and its E_2-degree is p, the merging does not break the $A_0|B_0$-biregularity requirement. We perform this merging for every vertex in U_1 and obtain an $A_0|B_0$-biregular graph of size $M|N$.

Case (b)*: When both K and L are not zero.* For this case, we first establish that $K \leqslant \delta(A_0, B_0)^2 N$ and $L \leqslant \delta(A_0, B_0)^2 M$, which will be used to bound the number of vertices in the phantom partition that are merged with the same vertex in the "real partition."

By (5.2) and (5.3), we have

$$0 < pK = a_1 M - b_1 N \leqslant a_1 M - N, \tag{5.4}$$

$$0 < pL = b_2 N - a_2 M \leqslant b_2 N - M. \tag{5.5}$$

Note that (5.4) implies $N < a_1 M$. Thus, plugging it into (5.5), we obtain

$$pL \leqslant b_2 N - M \leqslant b_2 a_1 M - M \leqslant b_2 a_1 M \leqslant \delta(A, B)^2 M.$$

Similarly, (5.5) implies $M < b_2 N$. Plugging it into (5.4), we obtain

$$pK \leqslant a_1 M - N \leqslant a_1 \cdot b_2 N - N = a_1 b_2 N \leqslant \delta(A, B)^2 N.$$

Hence

$$K \leqslant \delta(A_0, B_0)^2 N/p \qquad \text{and} \qquad L \leqslant \delta(A_0, B_0)^2 M/p. \tag{5.6}$$

Now, when we merge every vertex in the phantom partition with a vertex in the real partition, the bound $L \leqslant \delta(A_0, B_0)^2 M/p$ tells us that we can do it in such a way that every vertex in U_0 is merged with at most $\delta(A_0, B_0)^2/p$ vertices in U_1. Likewise, the bound $K \leqslant \delta(A_0, B_0)^2 N/p$ tells us that we can do the merging in such a way that each vertex in V_0 is merged with at most $\delta(A_0, B_0)^2/p$ vertices in V_1. After this merging we obtain an almost $A_0|B_0$-biregular graph $G = (U_0, V_0, E_1, E_2)$ with size $M|N$. Again the almost is because it is possible that there are parallel edges between two vertices in G. The bounds above have controlled the number of parallel edges that we need to worry about. We again perform the edge swapping to get rid of the parallel edges without affecting the degree of each vertex. Note that after the merging the total degree of each vertex increases by $\delta(A_0, B_0)^2$, since the degree of every vertex in $U_1 \cup V_1$ is p. The requirement that $M|N$ is extra-big-enough ensures that we have enough edges after the merging that we can perform the needed edge swapping to get rid of $\delta(A_0, B_0)^2$ parallel edges. ∎

5.3.2. Proof of Lemma 5.2 for extra-big-enough sizes. We now give the general construction for extra-big-enough sizes, extrapolating from the idea in the prior example. For simple degree matrices A and B with t rows, consider the formula $\Psi^2_{A|B}(\bar{x},\bar{y})$ given by

$$\exists z_{1,1}\cdots\exists z_{1,t}\ \exists z_{2,1}\cdots\exists z_{2,t}$$

$$\text{offset}(A)\cdot\bar{x}^T+\begin{pmatrix}\alpha_1 p z_{1,1}\\ \vdots\\ \alpha_t p z_{1,t}\end{pmatrix} = \text{offset}(B)\cdot\bar{y}^T+\begin{pmatrix}\beta_1 p z_{2,1}\\ \vdots\\ \beta_t p z_{2,t}\end{pmatrix}, \tag{5.7}$$

where $\alpha_i=1$ if row i in A consists of periodic entries and is 0 otherwise, and similarly $\beta_i=1$ if row i in B consists of periodic entries and is 0 otherwise.

This is again an edge counting equality, with the p multiples of $z_{1,i}$ and of $z_{2,i}$ representing additional edges due to the periodic factors. We can see that $\Psi^1_{A|B}(\bar{x},\bar{y})$ is a special case of it where all the constants $\alpha_1,\ldots,\alpha_t,\beta_1,\ldots,\beta_t$ are zero. We will show that $\Psi^2_{A|B}(\bar{x},\bar{y})$ captures all possible extra-big-enough sizes $\bar{M}|\bar{N}$ of $A|B$-biregular graphs, as formally stated in Lemma 5.9.

Lemma 5.9. *For each pair of simple degree matrices A, B and for each pair of size vectors $\bar{M}$, $\bar{N}$ such that $\bar{M}|\bar{N}$ is extra-big-enough for $A|B$, the formula $\Psi^2_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}$ if and only if there is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$.*

Proof. Let A and B be simple degree matrices with t rows. Let $\bar{M}|\bar{N}$ be extra-big-enough for $A|B$.

The if direction is just an edge counting equation for each color. Suppose there is an $A|B$-biregular graph $G=(U,V,E_1,\ldots,E_t)$ with size $\bar{M}|\bar{N}$. For each $i\in[t]$, the number of E_i-edges is the sum of E_i-degrees of vertices in U which is $\text{offset}(A_{1,*})\cdot\bar{M}+\alpha_i p z_{1,i}$ for some integer $z_{1,i}\geqslant 0$. This, of course, must equal the sum of E_i-degrees of vertices in V, and this is $\text{offset}(B_{1,*})\cdot\bar{N}+\beta_i p z_{2,i}$ for some integer $z_{2,i}\geqslant 0$. Thus, $\Psi^2_{A|B}(\bar{M},\bar{N})$ holds.

We now prove the only if direction. Suppose $\Psi^2_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}$. Abusing notation as before, we denote the values assigned to the variables $z_{i,j}$'s by the variables $z_{i,j}$'s themselves.

We are going to construct an $A|B$-biregular graph with size $\bar{M}|\bar{N}$. There are two cases—analogous to Cases (a) and (b) in subsection 5.3.1.

Case 1: $\alpha_i z_{1,i}\geqslant\beta_i z_{2,i}$ for every $i\in[t]$.

This case is analogous to scenarios (S2) and (S3) in the 1-color case, where we first "move a multiple of the period entries to one side" (S3) and "create a phantom partition for the period, then merge" (S2). It is also analogous to case (a) in Lemma 5.8. First, as in (S3), we move all the multiple of the period entries to one side—that is, rewrite (5.7) as

$$\text{offset}(A)\cdot\bar{M}^T+\begin{pmatrix}(\alpha_1 z_{1,1}-\beta_1 z_{2,1})p\\ \vdots\\ (\alpha_t z_{1,t}-\beta_t z_{2,t})p\end{pmatrix} = \text{offset}(B)\cdot\bar{N}^T.$$

We further rewrite the left-hand side as

$$(\text{offset}(A),pI_t)\cdot\begin{pmatrix}\bar{M}^T\\ \alpha_1 z_{1,1}-\beta_1 z_{2,1}\\ \vdots\\ \alpha_t z_{1,t}-\beta_t z_{2,t}\end{pmatrix} = \text{offset}(B)\cdot\bar{N}^T.$$

Recall that I_t is the identity matrix with size $t \times t$ and that $(\text{offset}(A), pI_t)$ denotes the matrix where the first sequence of columns are offset(A) and the next sequence of columns are pI_t. Intuitively, the submatrix pI_t represents p phantom partitions, each containing vertices whose E_i-degree is p on exactly one color i, with the other degrees being 0. The vector $(\alpha_1 z_{1,1} - \beta_1 z_{2,1}, \ldots, \alpha_t z_{1,t} - \beta_t z_{2,t})$ represents the sizes of these phantom partitions. Note that this is similar to (S2) in the 1-color case in Lemma 4.4, except that now we have one phantom partition for each color.

Let $C = (\text{offset}(A), pI_t)$ and $\bar{K} = (\alpha_1 z_{1,1} - \beta_1 z_{2,1}, \ldots, \alpha_t z_{1,t} - \beta_t z_{2,t})$. Since $\bar{M}|\bar{N}$ is extra-big-enough for $A|B$, $(\bar{M}, \bar{K})|\bar{N}$ is big-enough for $C|\text{offset}(B)$.

Note that C and offset(B) contain only fixed entries. By Lemma 5.6, there is an $(\text{offset}(A), pI_t)|\text{offset}(B)$-biregular graph $G = (U, V, E_1, \ldots, E_t)$ with size $(\bar{M}, \bar{K})|\bar{N}$.

Let $U = U_1 \uplus \cdots \uplus U_m \uplus W_1 \uplus \cdots \uplus W_t$ be its witness partition—that is, for every $i \in [t]$

- for every $j \in [m]$, the E_i-degree of every vertex in U_j is offset$(A_{i,j})$, and $|U_j| = M_j$;
- the E_i-degree of every vertex in W_i is p and $|W_i| = \alpha_i z_{1,i} - \beta_i z_{2,i}$, and for every $i' \neq i$, the $E_{i'}$-degree of every vertex in W_i is 0.

Here we actually "create phantom partitions $W_1, \ldots, W_t$ for the periods."

Observe that if $W_i \neq \emptyset$, i.e., $\alpha_i z_{1,i} - \beta_i z_{2,i} \neq 0$, then $\alpha_i \neq 0$. Since A is a simple matrix, its row i consists of only periodic entries, hence $\|\bar{M}^T\|_{\text{per}(A_{i,*})} = \|\bar{M}^T\|$. Since the sizes are extra-big-enough, we have $|U_1 \uplus \cdots \uplus U_m| = \|\bar{M}^T\| \geqslant \delta(A,B)^2 + 1$, where $\delta(A,B) = \max(\|\text{offset}(A)\|, \|\text{offset}(B)\|, p)$. For such an i, we are going to merge vertices in W_i with vertices in $U_1 \uplus \cdots \uplus U_m$—analogously to Lemma 4.4.

Let w be a vertex in W_i, where $W_j \neq \emptyset$. The number of vertices in G reachable by w in distance 2 (with any edges) is at most $\delta(A,B)^2$. Since $|U_1 \uplus \cdots \uplus U_m| \geqslant \delta^2 + 1$, there is a vertex $u \in U_1 \uplus \cdots \uplus U_m$ which is not reachable from w in distance 2. We can merge w with u. We perform such merging for every vertex in W_i. Since the E_i-degree of every vertex in W_i is p, and the $E_{i'}$-degree of vertices in W_i is 0, for every $i' \neq i$, such merging only increases the E_i-degree of a vertex in U by p. We continue in this way for every i where $W_i \neq \emptyset$, resulting in an $A|B$-biregular graph with size $\bar{M}|\bar{N}$.

The case where $\beta_i z_{2,i} \geqslant \alpha_i z_{1,i}$, for every $i \in [t]$, can be handled symmetrically.

Case 2: There are $i, i' \in [t]$ such that $\alpha_i z_{1,i} > \beta_i z_{2,i}$ and $\alpha_{i'} z_{1,i'} < \beta_{i'} z_{2,i'}$.

This case is analogous to case (b) in Lemma 5.8. Let Γ_1 be the set of indexes i such that $\alpha_i z_{1,i} > \beta_i z_{2,i}$ and Γ_2 be the set of indexes i such that $\alpha_i z_{1,i} < \beta_i z_{2,i}$. Since A and B are simple matrices, this means

- for every $i \in \Gamma_1$, $\alpha_i \neq 0$, i.e., row i in A consists of only periodic entries;
- likewise, for every $i \in \Gamma_2$, $\beta_i \neq 0$, i.e., row i in B consists of only periodic entries.

First, we can rewrite (5.7) as

$$\text{offset}(A) \cdot \bar{M}^T + \begin{pmatrix} pK_1 \\ \vdots \\ pK_t \end{pmatrix} = \text{offset}(B) \cdot \bar{N}^T + \begin{pmatrix} pL_1 \\ \vdots \\ pL_t \end{pmatrix},$$

where each K_i and L_i is defined as

$$K_i := \begin{cases} \alpha_i z_{1,i} - \beta_i z_{2,i} & \text{if } i \in \Gamma_1, \\ 0 & \text{if } i \notin \Gamma_1, \end{cases}$$

$$L_i := \begin{cases} \beta_i z_{2,i} - \alpha_i z_{1,i} & \text{if } i \in \Gamma_2, \\ 0 & \text{if } i \notin \Gamma_2. \end{cases}$$

We can further rewrite the formula:

$$(5.8)\qquad (\text{offset}(A), pI_t) \cdot \begin{pmatrix} \bar{M}^T \\ K_1 \\ \vdots \\ K_t \end{pmatrix} = (\text{offset}(B), pI_t) \cdot \begin{pmatrix} \bar{N}^T \\ L_1 \\ \vdots \\ L_t \end{pmatrix}.$$

In the following we let $C = (\text{offset}(A), pI_t)$ and $D = (\text{offset}(B), pI_t)$. We also let $\bar{K} = (K_1, \dots, K_t)$ and $\bar{L} = (L_1, \dots, L_t)$. Note that since $\bar{M}|\bar{N}$ is extra-big-enough for $A|B$, $(\bar{M}, \bar{K})|(\bar{N}, \bar{L})$ is big-enough for $C|D$. By Lemma 5.6, there is $C|D$-biregular graph $G = (U, V, E_1, \dots, E_t)$ with size $(\bar{M}, \bar{K})|(\bar{N}, \bar{L})$. We let $U = U_{0,1} \uplus \cdots \uplus U_{0,m} \uplus U_{1,1} \uplus \cdots \uplus U_{1,t}$ and $V = V_{0,1} \uplus \cdots \uplus V_{0,n} \uplus V_{1,1} \uplus \cdots \uplus V_{1,t}$ be the witness partition, where

- $\bar{M} = (|U_{0,1}|, \dots, |U_{0,m}|)$ and $\bar{K} = (|U_{1,1}|, \dots, |U_{1,t}|)$, and
- $\bar{N} = (|V_{0,1}|, \dots, |V_{0,n}|)$ and $\bar{L} = (|V_{1,1}|, \dots, |V_{1,t}|)$.

The partitions $U_{1,1}, \dots, U_{1,t}, V_{1,1}, \dots, V_{1,t}$ are the phantom partitions whose vertices are to be merged with the vertices in the real partitions $U_{0,1}, \dots, U_{0,m}, V_{0,1}, \dots, V_{0,n}$.

Similarly to case (b) in Lemma 5.8, we can bound the value of each K_i and L_i. For simplicity, we may first assume the following assumptions (a1) and (a2) hold.

(a1) For every $i \in \Gamma_1$, $A_{i,*}$ does not contain a 0^{+p} entry or, equivalently, $\text{offset}(A_{i,*})$ does not contain a zero entry.

(a2) Likewise, for every $i \in \Gamma_2$, $B_{i,*}$ does not contain a 0^{+p} entry.

Note that for every $i \in \Gamma_1$ we have

$$(5.9)\qquad \begin{aligned} 0 < pK_i = p(\alpha_i z_{1,i} - \beta_i z_{2,i}) &= \text{offset}(B_{i,*}) \cdot \bar{N}^T - \text{offset}(A_{i,*}) \cdot \bar{M}^T \\ &\leqslant \delta(A, B)\|\bar{N}^T\| - \|\bar{M}^T\|. \end{aligned}$$

In the last inequality we use the assumption that $\text{offset}(A_{i,*})$ does not contain a 0 entry. Similarly, for every $i \in \Gamma_2$ we have

$$(5.10)\qquad 0 < pL_i = p(\beta_i z_{2,i} - \alpha_i z_{1,i}) \leqslant \delta(A, B)\|\bar{M}^T\| - \|\bar{N}^T\|.$$

From (5.10), we obtain $\|\bar{N}^T\| \leqslant \delta(A, B)\|\bar{M}^T\|$. If we plug this into (5.9), we obtain that, for every $i \in \Gamma_1$,

$$(5.11)\qquad pK_i \leqslant \delta(A, B)^2\|\bar{M}^T\| - \|\bar{M}^T\| \leqslant \delta(A, B)^2\|\bar{M}^T\|.$$

Symmetrically, for every $i \in \Gamma_2$, we have

$$(5.12)\qquad pL_i \leqslant \delta(A, B)^2\|\bar{N}^T\|.$$

Inequalities (5.11) state that, for every $i \in \Gamma_1$, when performing the merging between vertices in the phantom partition $U_{1,i}$ and the real partitions $U_{0,1} \uplus \cdots \uplus U_{0,m}$, we can do so in such a way that every vertex in the real partition is merged with at most $\delta(A, B)^2/p$ vertices in the phantom partition. Likewise, (5.11) states similarly for $i \in \Gamma_2$ for the merging between vertices in the phantom partitions $V_{1,i}$ and the real partitions $V_{0,1} \uplus \cdots \uplus V_{0,m}$.

Now we reason as in the illustrative case. After the merging, we obtain an almost $A|B$-biregular graph with size $\bar{M}|\bar{N}$. As in the example, almost is because it is possible that there are parallel edges between two vertices in G and the established bounds above have controlled the number of parallel edges that we need to worry about. After the merging the total degree of each vertex increases by $t\delta(A_0, B_0)^2$. We perform the edge swapping to get rid of the parallel edges without affecting the degree of each vertex. The requirement that $\bar{M}|\bar{N}$ is extra-big-enough ensures that we have enough edges to perform the edge swapping. This completes the proof for case 2 when the assumptions (a1) and (a2) hold.

Now we consider the case when at least one of the assumptions (a1) or (a2) does not hold. The main idea is to rewrite the 0^{+p} entries in A and B as p^{+p} in such a way that the bounds in (5.11) and (5.12) still hold.

First, we rewrite (5.8),

$$(\text{offset}(A'), pI_t) \cdot \begin{pmatrix} \bar{M}^T \\ K_1' \\ \vdots \\ K_t' \end{pmatrix} = (\text{offset}(B'), pI_t) \cdot \begin{pmatrix} \bar{N}^T \\ L_1' \\ \vdots \\ L_t' \end{pmatrix},$$

where the matrix A' and the integers $K_1', \dots, K_t'$ are

(1) for every $i \notin \Gamma_1$, we let $A'_{i,*} = A_{i,*}$ and $K_i' = K_i$;

(2) for every $i \in \Gamma_1$ such that $A_{i,*}$ does not contain 0^{+p} entries, the row $A'_{i,*}$ is $A_{i,*}$ and $K_i' = K_i$;

(3) for every $i \in \Gamma_1$ such that $A_{i,*}$ contains 0^{+p} entries, we let $X = \{j : A_{i,j} = 0^{+p}\}$; moreover,

 (3.a) if $K_i < \|\bar{M}^T\|_X$, then $A'_{i,*} = A_{i,*}$ and $K_i' = K_i$, and

 (3.b) if $K_i \geqslant \|\bar{M}^T\|_X$, then $A'_{i,*}$ is obtained from $A_{i,*}$ by changing every 0^{+p} entry with p^{+p} and $K_i' = K_i - \|\bar{M}^T\|_X$.

The matrix B' and the integers $L_1', \dots, L_t'$ are defined in a similar manner.

(4) For every $i \notin \Gamma_2$, we let $B'_{i,*} = B_{i,*}$ and $L_i' = L_i$.

(5) For every $i \in \Gamma_2$ such that $B_{i,*}$ does not contain 0^{+p} entries, the row $B'_{i,*}$ is $B_{i,*}$ and $L_i' = L_i$.

(6) For every $i \in \Gamma_2$ such that $B_{i,*}$ contains 0^{+p} entries, we let $X = \{j : B_{i,j} = 0^{+p}\}$; moreover,

 (6.a) if $L_i < \|\bar{N}^T\|_X$, then $B'_{i,*} = B_{i,*}$ and $L_i' = L_i$, and

 (6.b) if $L_i \geqslant \|\bar{N}^T\|_X$, then $B'_{i,*}$ is obtained from $B_{i,*}$ by changing every 0^{+p} entry with p^{+p} and $L_i' = L_i - \|\bar{N}^T\|_X$.

Note that the only difference between A and A', and between B and B', is in (3.b) and (6.b), respectively, where some 0^{+p} entries are changed into p^{+p}. Thus, an $A'|B'$-biregular graph is also an $A|B$-biregular graph.

Performing a similar calculation as in (5.9)–(5.12), we can show that

- for every $i \in \Gamma_1$, $K_i' \leqslant \delta(A, B)^2 \|\bar{M}^T\|/p$;
- for every $i \in \Gamma_2$, $L_i' \leqslant \delta(A, B)^2 \|\bar{N}^T\|/p$.

The construction of an $A'|B'$-biregular graph with size $\bar{M}|\bar{N}$ can be done almost verbatim as above. □

Remark 5.10. It is only in case 2 in the proof of Lemma 5.9 that we require the quantity

$$\max(\|\bar{M}^T\|_{\text{nz}(A_{i,*})}, \|\bar{N}^T\|_{\text{nz}(B_{i,*})}),$$

which is precisely the number of vertices with nonzero E_i-degree in an $A|B$-biregular graph of size $\bar{M}|\bar{N}$, to be at least quartic in $\delta(A, B)$, and not quadratic as in section 4. This is because the total degree of each vertex increases by at most $t\delta(A, B)^2$ after the merging between the vertices in the phantom partitions and real partitions.

Thus, we require that the number of edges is at least quartic in $\delta(A,B)$ to ensure there are enough edges to perform edge swapping to get rid of the parallel edges. Note also that the restriction of A and B to simple matrices allows us to merge every vertex in the phantom partition with any vertex in the real partition. Thus we can perform the merging in such a way that the total degree of each vertex in the real partition increases by at most $t\delta(A,B)^2$.

5.4. Encoding of not big/extra-big-enough components for simple matrices. Lemma 5.9 gives a formula that captures the existence of biregular graphs for extra-big-enough sizes for simple degree matrices. We now turn to sizes that are not big/extra-big-enough. Here we will use the same idea of fixed size encoding as in the 1-color case.

Note that a "not-extra-big-enough size" means that one of the conditions (a)–(c) in Definition 5.4 is violated and thus some of the entries in the size vectors $\bar{M}, \bar{N}$ are already fixed. For example, if condition (a) is violated, then $\max(\|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})}, \|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})})$ is between 1 and $8t^2\delta(A,B)^4$ for some $i \in [t]$. So in this case we can fix the values of $\|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})}$ and $\|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})}$ as some r_1 and r_2, where r_1, r_2 are in-between 1 and $8t^2\delta(A,B)^4$. As in the 1-color case (Lemma 4.7), the idea will be that a fixed number of nonzero degree vertices in a graph can be encoded as formulas, along the lines of subsection 4.1.3.

We detail the formula construction for the case where for some color, condition (a) is violated, but conditions (b) and (c) hold. All the other cases can be handled in a similar manner. We fix degree matrices A and B with t rows, and let m and n be the number of columns in A and B. For simplicity, we focus on the case where the color where (a) is violated is the tth row. For integers $r_1, r_2 \geqslant 0$, we define a formula $\Phi^{r_1,r_2}_{A|B}(\bar{x},\bar{y})$ that captures precisely the sizes $\bar{M}|\bar{N}$ of $A|B$-biregular graph where $\|\bar{M}^T\|_{\mathrm{nz}(A_{t,*})} = r_1$ and $\|\bar{N}^T\|_{\mathrm{nz}(B_{t,*})} = r_2$. The construction is by induction on $r_1 + r_2$ and the number of rows in the degree matrices A and B.

- When the number of rows in A, B is 1, the formula $\Phi^{i,r_1,r_2}_{A|B}(\bar{x},\bar{y})$ simply enumerates all possible sizes of $A|B$-biregular graphs.
  Such an enumeration is possible since the number of vertices with nonzero degree on the left=hand side is fixed to r_1, and the number of vertices with nonzero degree on the right-hand side is fixed to r_2.
- If the ith row in both A and B contains periodic entries, the formula $\Phi^{i,r_1,r_2}_{A|B}(\bar{x},\bar{y})$ simply enumerates all possible sizes of $A|B$-biregular graphs, where r_1 is the number of vertices on the left-hand side and r_2 is the number of vertices on the right-hand side.
  Here it is useful to recall that A (resp., B) is a simple matrix, hence either the entries in A (resp., B) are all fixed entries or are all periodic entries.
- When $r_1 + r_2 = 0$, we get the formula

$$\Phi^{r_1,r_2}_{A|B}(\bar{x},\bar{y}) := \Phi_{\tilde{A}|\tilde{B}}(\bar{x}_0,\bar{y}_0) \ \wedge\ \|\bar{x}^T\|_{\mathrm{nz}(A_{t,*})} = 0 \ \wedge\ \|\bar{y}^T\|_{\mathrm{nz}(B_{t,*})} = 0,$$

  where $\tilde{A}$ is the matrix A without the tth row and without the columns in $\mathrm{nz}(A_{t,*})$, $\tilde{B}$ is the matrix B without the tth row and without the columns in $\mathrm{nz}(B_{t,*})$, and $\bar{x}_0$ and $\bar{y}_0$ are the vectors $\bar{x}$ and $\bar{y}$ without the components in $\mathrm{nz}(A_{t,*})$ and $\mathrm{nz}(B_{t,*})$, respectively.
  The purpose of the formula $\Phi_{\tilde{A}|\tilde{B}}(\bar{x}_0,\bar{y}_0)$ is to capture all possible sizes of $\tilde{A}|\tilde{B}$-biregular graphs. Formally, it is defined as

$$\Psi^2_{\tilde{A}|\tilde{B}}(\bar{x}_0,\bar{y}_0) \ \vee\ \Theta_{\tilde{A}|\tilde{B}}(\bar{x}_0,\bar{y}_0),$$

where $\Psi^2_{\tilde{A}|\tilde{B}}(\bar{x}_0,\bar{y}_0)$ captures all the extra-big-enough sizes of $\tilde{A}|\tilde{B}$-biregular graphs as defined in subsection 5.3 and $\Theta_{\tilde{A}|\tilde{B}}(\bar{x}_0,\bar{y}_0)$ captures all the not-extra-big-enough sizes of $\tilde{A}|\tilde{B}$-biregular graphs. Note that the number of rows in $\tilde{A}|\tilde{B}$ is now $t-1$, hence the formula $\Theta_{\tilde{A}|\tilde{B}}(\bar{x}_0,\bar{y}_0)$ can be defined inductively. The intuition behind the matrices $\tilde{A}$ and $\tilde{B}$ is that, since $\|\bar{x}^T\|_{\mathrm{nz}(A_{t,*})} = r_1 = 0$ and $\|\bar{y}^T\|_{\mathrm{nz}(B_{t,*})} = r_2 = 0$, we can ignore the color t, i.e., by removing the tth row in A and B and all the corresponding columns in $\mathrm{nz}(A_{t,*})$ and $\mathrm{nz}(B_{t,*})$.

- When $r_1 + r_2 \geqslant 1$, at least one of r_1 or r_2 is bigger than or equal to 1. When $r_1 \geqslant 1$, we let

$$\Phi^{r_1,r_2}_{A|B}(\bar{x},\bar{y}) := \exists s_1 \cdots \exists s_t\ \exists \bar{z}_0 \exists \bar{z}_1 \cdots \exists \bar{z}_t \bigvee_{j \in \mathrm{nz}(A_{t,*})} \left(\begin{array}{l} (x_{1,j} \neq 0)\ \wedge\ \bar{y} = \sum_{\ell=0}^{t} \bar{z}_\ell \\ \wedge\ \bigwedge_{\ell=1}^{t} \|\bar{z}_\ell^T\| = \mathrm{offset}(A_{\ell,j}) + \alpha_\ell \cdot p \cdot s_\ell \\ \wedge\ \Phi^{r_1-1,r_2}_{A|(B,B-J_1,\ldots,B-J_t)}(\bar{x} - \mathbf{e}_j, \bar{z}_0, \bar{z}_1, \ldots, \bar{z}_t) \end{array} \right),$$

where each α_ℓ is in $\{0,1\}$ with $\alpha_\ell = 1$ if and only if $A_{\ell,j}$ is a periodic entry; each J_ℓ is a matrix with size $(t \times m)$, where row ℓ consists of all 1 entries and all the other rows have only 0 entries.

When $r_2 \geqslant 1$, the formula can be defined symmetrically with the roles of $A, \bar{x}$ and $B, \bar{y}$ being swapped.

The following lemma states the correctness of the formula constructed above.

LEMMA 5.11. *For every pair of simple degree matrices A, B with t rows, for all integers $r_1, r_2 \geqslant 0$, for all size vectors $\bar{M}, \bar{N}$, the formula $\Phi^{r_1,r_2}_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}$ if and only if there is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$, where $\|\bar{M}^T\|_{\mathrm{nz}(A_{t,*})} = r_1$ and $\|\bar{N}^T\|_{\mathrm{nz}(B_{t,*})} = r_2$.*

The proof of Lemma 5.11 is a straightforward generalization of Lemma 4.8, hence we omit it.

The case where (b) or (c) is violated for some color $i \in [t]$ can be treated in a similar manner. Note that in the case when *both* (b) and (c) are violated, i.e., $1 \leqslant \|\bar{M}^T\|_{\mathrm{per}(A_{i_1,*})} \leqslant \delta(A,B)^2$ and $1 \leqslant \|\bar{N}^T\|_{\mathrm{per}(B_{i_2,*})} \leqslant \delta(A,B)^2$ for some $i_1, i_2 \in [t]$, the number of vertices is fixed to some r in-between 1 and $2\delta(A,B)^2$, since $\mathrm{per}_{A_{i_1,*}} = [m]$ and $\mathrm{per}(B_{i_2,*}) = [n]$ due to A and B being simple matrices. Thus, in this case all possible sizes of $A|B$-biregular graphs can simply be enumerated.

Remark 5.12. The following observations about the formula will be useful in our complexity analysis later on. By pulling out the disjunction, we can rewrite the formula $\Phi^{r_1,r_2}_{A|B}(\bar{x},\bar{y})$ as a disjunction $\bigvee_i \varphi_i$ conjoined with $\Phi_{\tilde{A}|\tilde{B}}(\bar{x}_0,\bar{y}_0)$, where each φ_i is a conjunction of $O(t(r_1+r_2))$ (in)equations. Since r_1, r_2 ranges between 1 and $\max(8t^2\delta(A,B)^4, t\delta(A,B)) = 8t^2\delta(A,B)^4$, each φ_i is a conjunction of $O(t^3\delta(A,B)^4)$ (in)equations conjoined with $\Phi_{\tilde{A}|\tilde{B}}(\bar{x}_0,\bar{y})$. It is useful to recall that $\tilde{A}|\tilde{B}$ now have one less row than $A|B$.

By straightforward induction on the number of rows t, we observe that the formula $\Phi^{r_1,r_2}_{A|B}(\bar{x},\bar{y})$ can be written as a disjunction $\bigvee_i \varphi_i$, where each φ_i is a conjunction of $O(t^4\delta(A,B)^4)$ (in)equations.

5.5. Proof of Lemma 5.2. To wrap up this section, for simple matrices A and B, we define the formula $\mathsf{bireg}_{A|B}(\bar{x},\bar{y})$ required in Lemma 5.2 to characterize all the possible sizes of $A|B$-biregular graph, without the completeness requirement,

$$\mathsf{bireg}_{A|B}(\bar{x},\bar{y}) := \Psi^2_{A|B}(\bar{x},\bar{y}) \ \vee \ \bigvee_{i\in[\ell]} \Phi_i(\bar{x},\bar{y}),$$

where $\Psi^2_{A|B}(\bar{x},\bar{y})$ is defined in (5.7) to deal with the big-enough sizes, while the disjunction $\bigvee_{i\in[\ell]} \Phi_i(\bar{x},\bar{y})$ deals with the not-extra-big-enough sizes as defined in subsection 5.4. Here we assume an enumeration of all the formulas $\Phi_1(\bar{x},\bar{y}),\ldots,\Phi_\ell(\bar{x},\bar{y})$ that deal with the not-extra-big-enough sizes. The correctness of the construction follows immediately from Lemmas 5.9 and 5.11.

Remark 5.13. Let t be the number of rows in matrices A and B and let m and n be the number of columns in A and B, respectively. By Remark 5.12, each $\Phi_i(\bar{x},\bar{y})$ is a disjunction of conjunctions of $O(t^4\delta(A,B)^4)$ (in)equations. Since $\Psi^2_{A|B}(\bar{x},\bar{y})$ is a conjunction of t equations, the formula $\mathsf{bireg}_{A|B}(\bar{x},\bar{y})$ can be written as a disjunction $\bigvee_i \varphi_i$, where each φ_i is a conjunction of $O(t^4\delta(A,B)^4)$ (in)equations.

6. Proof of Theorem 3.2 for the case of simple matrices with the completeness requirement being enforced. We will now consider the formula defining possible partition sizes, still restricting to simple biregular graphs, but now enforcing the completeness restriction. This will be done via reduction to the case where the completeness restriction has not been enforced.

We introduce a further restriction on the matrices that will be useful.

DEFINITION 6.1. *For a pair of simple matrices $A|B$ (with the same number of rows), we say that $A|B$ is* a good pair *if there is i such that row i is periodic in both A and B.*

Remark 6.2. Note that if $A|B$ is not a good pair, then complete $A|B$-biregular graphs can only have up to $2\delta(A,B)$ vertices. Indeed, suppose $G=(U,V,E_1,\ldots,E_t)$ is a complete $A|B$-biregular graph. Since $A|B$ is not a good pair, for every $i\in[t]$, the number of edges in E_i is at most $\delta(A,B)|U|$ or $\delta(A,B)|V|$. Thus, $\sum_{i\in[t]}|E_i|$ is at most $\delta(A,B)(|U|+|V|)$. On the other hand, the fact that G is complete implies that $\sum_{i\in[t]}|E_i| = |U||V|$ which is strictly bigger than $\delta(A,B)(|U|+|V|)$ when $|U|+|V| > 2\delta(A,B)$. So, when $A|B$ is not a good pair, to capture all possible sizes of complete $A|B$-biregular graphs, we simply write a formula that enumerates all possible $\bar{M}|\bar{N}$, where $\|\bar{M}^T\|+\|\bar{N}^T\| \leqslant 2\delta(A,B)$.

So it suffices to define the formula that captures all possible sizes of complete (finite) $A|B$-biregular graphs where A and B are both simple matrices and $A|B$ is a good pair. Let $\bar{x}=(x_1,\ldots,x_m)$ and $\bar{y}=(y_1,\ldots,y_n)$. Let $A\in\mathbb{N}^{t\times m}_{+p}$ and $B\in\mathbb{N}^{t\times n}_{+p}$ be simple matrices such that $A|B$ is a good pair. Let $\xi_{A|B}(\bar{x},\bar{y})$ be the formula

$$\mathsf{bireg}_{A|B}(\bar{x},\bar{y}) \tag{6.1}$$

$$\wedge \quad \bigwedge_{j\in[m]} x_j\neq 0 \ \rightarrow\ \exists z\ \|\bar{y}^T\| = \|\mathrm{offset}(A_{*,j})\| + pz \tag{6.2}$$

$$\wedge \quad \bigwedge_{j\in[n]} y_j\neq 0 \ \rightarrow\ \exists z\ \|\bar{x}^T\| = \|\mathrm{offset}(B_{*,j})\| + pz. \tag{6.3}$$

Here $\mathsf{bireg}_{A|B}(\bar{x},\bar{y})$ is the formula characterizing the situation without the completeness requirement.

Intuitively, (6.2) states that the number of vertices on the right-hand side must equal the total degree of the vertices on the left-hand side. Likewise, (6.3) states that the number of vertices on the left-hand side must equal the total degree of the vertices on the right-hand side.

LEMMA 6.3. *For every pair of simple matrices A and B such that $A|B$ is a good pair, for every pair of size vectors $\bar{M}$ and $\bar{N}$, the formula $\xi_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}$ exactly when there is a complete $A|B$-biregular graph of size $\bar{M}|\bar{N}$.*

Proof. That $\xi_{A|B}(\bar{M},\bar{N})$ is a necessary condition for the existence of a complete $A|B$-biregular graph is straightforward. This follows from the fact that if $G = (U, V, E_1, \dots, E_t)$ is a complete $A|B$-biregular graph then the sum of all E_i-degrees of every vertex in U must equal $|V|$ and, likewise, the sum of all E_i-degrees of every vertex in V must equal $|U|$.

Now we show that it is also a sufficient condition. Suppose $\xi_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}$. Thus, $\mathsf{bireg}_{A|B}(\bar{M},\bar{N})$ holds, and by Lemma 5.2, there is a (not necessarily complete) $A|B$-biregular graph $G = (U, V, E_1, \dots, E_t)$ with size $\bar{M}|\bar{N}$. We will show how to make G complete.

Let $U = U_1 \uplus \cdots \uplus U_m$ and $V = V_1 \uplus \cdots \uplus V_n$ be the witness partition. Since $A|B$ is a good pair, there is i_0 such that row i_0 is periodic in both A and B. Now, for every $(u,v) \notin E_1 \cup \cdots \cup E_t$, we define (u,v) to be in E_{i_0}. Obviously, after adding such E_{i_0}-edges, the graph G becomes complete. We argue that G is still $A|B$-biregular by showing that

(a) for every $j \in [m]$, for every vertex $w \in U_j$, the E_{i_0}-degree of w increases by a multiple of p;

(b) for every $j \in [n]$, for every vertex $w \in V_j$, the E_{i_0}-degree of w increases by a multiple of p.

We prove (a), fixing $w \in U_j$. The E_{i_0}-degree of w increases by

$$|V| - \sum_{i\in[t]} \deg_{E_i}(w).$$

Note that (6.2) forces $|V|$ to be

$$|V| = \|\mathrm{offset}(A_{*,j})\|^{+p} = \|\mathrm{offset}(A_{*,j})\| + (\text{some multiple of } p).$$

On the other hand, we also have

$$\sum_{i\in[t]} \deg_{E_i}(w) = \sum_{i\in[t]} A_{i,j} = \|\mathrm{offset}(A_{*,j})\| + (\text{some multiple of } p).$$

Here it is useful to recall that row i_0 in A contains periodic entries, hence the additional term “some multiple of p.” Thus, the quantity $|V| - \sum_{i\in[t]} \deg_{E_i}(w)$ is a multiple of p, and therefore the E_{i_0}-degree of w only increases by a multiple of p. This does not violate the $A|B$-biregularity condition.

Part (b) can be proven in a similar manner to (6.3). This completes our proof of Lemma 6.3. ∎

Remark 6.4. We will again make some further observations that will be important only for the complexity analysis, which will be detailed in section 8. For each $j \in [m]$, let $a_j = \|\mathrm{offset}(*,j)^T\|$. We first rewrite (6.2) as follows:

$$\bigvee_{j_1\in[m]} \left(\exists z \ \|\bar{y}^T\| = a_{j_1} + pz \ \wedge \ x_{j_1} \neq 0 \ \wedge \bigwedge_{j_2\in[m] \text{ s.t. } a_{j_2} \not\equiv a_{j_1} \bmod p} x_{j_2} = 0 \right).$$

Indeed, if $x_{j_1}, x_{j_2} \neq 0$, then $\|\bar{y}^T\| = a_{j_1}^{+p}$ and $\|\bar{y}^T\| = a_{j_2}^{+p}$, which implies $a_{j_1} \equiv a_{j_2} \bmod p$. Therefore, if $x_{j_1} \neq 0$, then $x_{j_2} = 0$ whenever $a_{j_2} \not\equiv a_{j_1} \bmod p$. We also rewrite (6.3) in a similar manner.

Note that (6.2) yields $O(m)$ equalities, while the rewriting above transforms it into a disjunction of $O(1)$ (in)equations.[7] By Remark 5.13, the formula $\xi_{A|B}(\bar{x},\bar{y})$ is a disjunction of conjunctions of $O(t^4\delta(A,B)^4)$ (in)equations, where t is the number of rows in matrices A and B.

To wrap up section 6, we define the formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{x},\bar{y})$ for simple matrices A and B as follows,

$$\mathsf{c\text{-}bireg}_{A|B}(\bar{x},\bar{y}) := \begin{cases} \xi_{A|B}(\bar{x},\bar{y}) & \text{if } A|B \text{ is a good pair,} \\ \bigvee_i \phi_i(\bar{x},\bar{y}) & \text{if } A|B \text{ is not a good pair,} \end{cases} \tag{6.4}$$

where $\xi_{A|B}(\bar{x},\bar{y})$ is defined in (6.1)–(6.3) when $A|B$ is a good pair and the disjunction $\bigvee_i \phi_i(\bar{x},\bar{y})$ enumerates all possible sizes $\bar{M}|\bar{N}$ when $A|B$ is not a good pair. Recall that by Remark 6.2, when $A|B$ is not a good pair, complete $A|B$-biregular graphs can only have sizes $\bar{M}|\bar{N}$, where $\|\bar{M}^T\| + \|\bar{N}^T\| \leqslant 2\delta(A,B)$. Since there are only finitely many such sizes, they can be enumerated. The correctness of the formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{x},\bar{y})$ follows immediately from Lemma 6.3 and Remark 6.2, as stated formally in Lemma 6.5.

Lemma 6.5. *For every pair of simple matrices A and B and for every pair of size vectors $\bar{M}$ and $\bar{N}$, the formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}$ exactly when there is a complete $A|B$-biregular graph of size $\bar{M}|\bar{N}$.*

7. Proof of Theorems 3.2 and 3.3. In this section we will present the proof of Theorems 3.2 and 3.3. Recall that Theorem 3.2 states that for all arbitrary degree matrices A and B, we can effectively construct a Presburger formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{x},\bar{y})$ that captures all possible sizes of complete $A|B$-biregular graphs. Theorem 3.3 is the analog for the directed graphs.

In section 6 we showed how to construct Presburger formulas that capture all possible sizes of complete simple $A|B$-biregular graphs, i.e., where the degree matrices A and B are simple matrices. In this section we will show how to reduce the nonsimple matrices to simple matrices for biregular graphs. We divide this section into three subsections. We begin with an example that shows the main idea in section 7.1. In section 7.2 we present the general reduction from nonsimple biregular graphs to simple biregular graphs. Finally, in section 7.3 we deal with the regular digraphs.

7.1. A special case illustrating the reduction. Consider the degree matrices $A_0 = (a_1, a_2^{+p})$ and $B_0 = (b_1, b_2^{+p})$, where a_1, a_2, b_1, b_2 are all nonzero integers. Note that this is just the 1-color case, which is already handled in section 4. The choice of 1 color is for the sake of simplicity. Obviously, they are not simple matrices, since each row contains both fixed and periodic entries. We will show that every $A_0|B_0$-biregular graph can be viewed as a collection of four simple biregular graphs, as stated formally in Theorem 7.1. Note that this example has only one color, and we already know how to construct the required Presburger formula from section 4. The purpose of this section is only to illustrate the reduction from nonsimple matrices to simple matrices.

The main idea is as follows. Suppose we have $A_0|B_0$-biregular graph $G = (U,V,E)$ with witness partition $U = U_1 \uplus U_2$ and $V = V_1 \uplus V_2$. We will decompose the graph into 4 induced bipartite subgraphs, each representing the restriction to one partition on

[7] Here we do not count equations of the form $x = 0$ since such a variable x can be ignored during the computation, thus, becomes negligible in the complexity analysis.

the left and one on the right.[8] We will show below that each such subgraph satisfies a biregularity condition:

- The induced subgraph $G[U_1 \cup V_1]$ is a $(0,1,\ldots,a_1)|(0,1,\ldots,b_1)$-biregular graph.
- The induced subgraph $G[U_1 \cup V_2]$ is an $(a_1, a_1-1,\ldots,0)|(0^{+p},1^{+p},\ldots,b_2^{+p})$-biregular graph.
- The induced subgraph $G[U_2 \cup V_1]$ is a $(0^{+p},1^{+p},\ldots,a_2^{+p})|(b_1,b_1-1,\ldots,0)$-biregular graph.
- The induced subgraph $G[U_2 \cup V_2]$ is an $(a_2^{+p},(a_2-1)^{+p},\ldots,0^{+p})|(b_2^{+p},(b_2-1)^{+p},\ldots,0^{+p})$-biregular graph.

Note that the degree matrices involved are all simple matrices. For example, the degree matrix $(0,1,\ldots,a_1)$, which has only one row, is simple, since every row contains only fixed entries. As another example, the degree matrix $(0^{+p},1^{+p},\ldots,a_2^{+p})$ is also simple, since every row contains only periodic entries.

We call the decomposition of G into the subgraphs $G[U_1\cup V_1]$, $G[U_1\cup V_2]$, $G[U_2\cup V_1]$, and $G[U_2\cup V_2]$ *the degree-based decomposition of* G. We reduce a characterization of sizes of $A_0|B_0$-biregular graphs to a characterization of the sizes of the components of the decomposition.

Theorem 7.1. *For every pair $M_1, M_2 \in \mathbb{N}^2$ and every pair $N_1, N_2 \in \mathbb{N}^2$, the following are equivalent.*

(a) *There is an $A_0|B_0$-biregular graph with size $(M_1,M_2)|(N_1,N_2)$.*

(b) *There exist size vectors $\bar{K}_1 \in \mathbb{N}^{a_1+1}$, $\bar{K}_2 \in \mathbb{N}^{a_2+1}$, $\bar{L}_1 \in \mathbb{N}^{b_1+1}$, $\bar{L}_2 \in \mathbb{N}^{b_2+1}$ such that $\|\bar{K}_1^T\| = M_1$, $\|\bar{K}_2^T\| = M_2$, $\|\bar{L}_1^T\| = N_1$, and $\|\bar{L}_2^T\| = N_2$ and*
- *a $(0,1,\ldots,a_1)|(0,1,\ldots,b_1)$-biregular graph with size $\bar{K}_1|\bar{L}_1$;*
- *an $(a_1,a_1-1,\ldots,0)|(0^{+p},1^{+p},\ldots,b_2^{+p})$-biregular graph with size $\bar{K}_1|\bar{L}_2$;*
- *a $(0^{+p},1^{+p},\ldots,a_2^{+p})|(b_1,b_1-1,\ldots,0)$-biregular graph with size $\bar{K}_2|\bar{L}_1$;*
- *an $(a_2^{+p},(a_2-1)^{+p},\ldots,0^{+p})|(b_2^{+p},(b_2-1)^{+p},\ldots,0^{+p})$-biregular graph with size $\bar{K}_2|\bar{L}_2$,*

Note that there can be several vectors $\bar{K}_1\ldots$ satisfying the conditions on norms in the theorem. But the condition on sizes can clearly be described in Presburger arithmetic, so this allows us to get a Presburger formula for the sizes of an $A_0|B_0$-biregular graph, assuming we can get such a formula for the simple case.

The proof of Theorem 7.1 is conceptually simple, but rather technical. We divide it into two lemmas: Lemma 7.2 which implies the only if direction and Lemma 7.3 which deals with the if direction. Below we let $[0,k]$ denote the set $\{0,1,\ldots,k\}$ for an integer $k \geqslant 0$.

Lemma 7.2. *For every $A_0|B_0$-biregular graph $G=(U,V,E)$ with witness partition $U = U_1 \uplus U_2$ and $V = V_1 \uplus V_2$, there exist size vectors $\bar{K}_1 \in \mathbb{N}^{a_1+1}$, $\bar{K}_2 \in \mathbb{N}^{a_2+1}$, $\bar{L}_1 \in \mathbb{N}^{b_1+1}$, and $\bar{L}_2 \in \mathbb{N}^{b_2+1}$ such that*

1. *the induced subgraph $G[U_1\cup V_1]$ is a $(0,1,\ldots,a_1)|(0,1,\ldots,b_1)$-biregular graph with size $\bar{K}_1|\bar{L}_1$;*
2. *the induced subgraph $G[U_1\cup V_2]$ is an $(a_1,a_1-1,\ldots,0)|(0^{+p},1^{+p},\ldots,b_2^{+p})$-biregular graph with size $\bar{K}_1|\bar{L}_2$;*
3. *the induced subgraph $G[U_2\cup V_1]$ is a $(0^{+p},1^{+p},\ldots,a_2^{+p})|(b_1,b_1-1,\ldots,0)$-biregular graph with size $\bar{K}_2|\bar{L}_1$;*
4. *the induced subgraph $G[U_2\cup V_2]$ is an $(a_2^{+p},(a_2-1)^{+p},\ldots,0^{+p})|(b_2^{+p},(b_2-1)^{+p},\ldots,0^{+p})$-biregular graph with size $\bar{K}_2|\bar{L}_2$.*

[8]As usual, for a graph $G=(V,E)$ and for a subset $S \subseteq V$, the notation $G[S]$ denotes the subgraph induced in G by the set S.

Proof. Let $G=(U,V,E)$ be a $A_0|B_0$-biregular graph with size $(M_1,M_2)|(N_1,N_2)$. Let $U=U_1 \uplus U_2$ and $V=V_1 \uplus V_2$ be the witness partition, where

- every vertex in U_1 has degree a_1 and every vertex in U_2 has degree a_2^{+p};
- every vertex in V_1 has degree b_1 and every vertex in U_2 has degree b_2^{+p}.

We partition the set U_1 as follows,

$$U_1 \;=\; U_{1,0} \uplus U_{1,1} \uplus \cdots \uplus U_{1,a_1},$$

where for each $j\in[0,a_1]$, the set $U_{1,j}$ is the set of vertices in U_1 with j neighbors in V_1 and (a_1-j) neighbors in V_2. See Figure 6 for an illustration. We repartition the set U_2, V_1, V_2 in a similar manner.

- Let $U_2=U_{2,0}\uplus U_{2,1}\uplus\cdots\uplus U_{2,a_2}$, where for each $j\in[0,a_2]$, $U_{2,j}$ is the set of vertices in U_2 with j^{+p} neighbors in V_1 and $(a_2-j)^{+p}$ neighbors in V_2.
- We let $V_1=V_{1,0}\uplus V_{1,1}\uplus\cdots\uplus V_{1,b_1}$, where for each $j\in[0,b_1]$, $V_{1,j}$ is the set of vertices in V_1 with j neighbors in U_1 and (b_1-j) neighbors in U_2.
- We let $V_2=V_{2,0}\uplus V_{2,1}\uplus\cdots\uplus V_{2,b_2}$, where for each $j\in[0,b_2]$, $V_{2,j}$ is the set of vertices in V_2 iwth j^{+p} neighbors in U_1 and $(b_2-j)^{+p}$ neighbors in U_2.

Now, we let $\bar{K}_1,\bar{K}_2,\bar{L}_1,\bar{L}_2$ as follows:

$$\begin{aligned}
\bar{K}_1 &:= (|U_{1,0}|,|U_{1,1}|,\dots,|U_{1,a_1}|), & \bar{K}_2 &:= (|U_{2,0}|,|U_{2,1}|,\dots,|U_{2,a_2}|),\\
\bar{L}_1 &:= (|V_{1,0}|,|V_{1,1}|,\dots,|V_{1,b_1}|), & \bar{L}_2 &:= (|V_{2,0}|,|V_{2,1}|,\dots,|V_{2,b_2}|).
\end{aligned}$$

To complete the proof of Lemma 7.2, we show

(1) $G[U_1\cup V_1]$ is a $(0,1,\dots,a_1)|(0,1,\dots,b_1)$-biregular graph with size $\bar{K}_1|\bar{L}_1$;

(2) $G[U_1\cup V_2]$ is an $(a_1,a_1-1,\dots,0)|(0^{+p},1^{+p},\dots,b_2^{+p})$-biregular graph with size $\bar{K}_1|\bar{L}_2$;

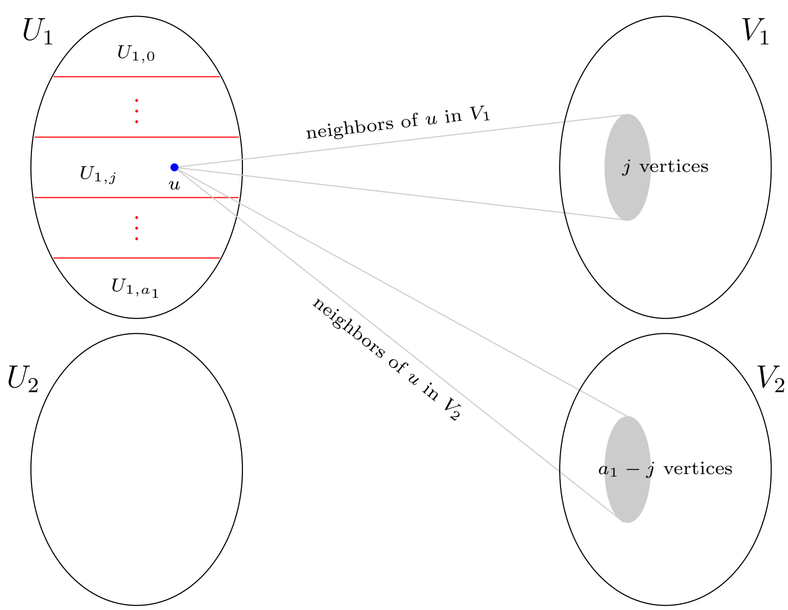

FIG. 6. *An illustration for the proof of Lemma 7.2. $G=(U,V,E)$ is an $A_0|B_0$-biregular graph with $U=U_1\uplus U_2$ and $V=V_1\uplus V_2$ the witness partition. We partition $U_1=U_{1,0}\uplus\cdots\uplus U_{1,a_1}$, where for each $j\in[0,a_1]$, each vertex $u\in U_{1,j}$ has j neighbors in V_1 and (a_1-j) neighbors in V_2. Similarly we partition $U_2=U_{2,0}\uplus\cdots\uplus U_{2,a_2}$, $V_1=V_{1,0}\uplus\cdots\uplus V_{1,b_1}$, and $V_2=V_{2,0}\uplus\cdots\uplus V_{2,b_2}$. Note: color appears only in the online article.*

(3) $G[U_2 \cup V_1]$ is a $(0^{+p}, 1^{+p}, \ldots, a_2^{+p})|(b_1, b_1-1, \ldots, 0)$-biregular graph with size $\bar{K}_2|\bar{L}_1$;
(4) $G[U_2 \cup V_2]$ is an $(a_2^{+p}, (a_2-1)^{+p}, \ldots, 0^{+p})|(b_2^{+p}, (b_2-1)^{+p}, \ldots, 0^{+p})$-biregular graph with size $\bar{K}_2|\bar{L}_2$.

To prove (1), note that

- for each $j_1 \in [0, a_1]$, each vertex in U_{1,j_1} has degree j_1 in $G[U_1 \cup V_1]$;
- for each $j_2 \in [0, b_1]$, each vertex in V_{1,j_2} has degree j_2 in $G[U_1 \cup V_1]$.

Thus, $U_1 = U_{1,0} \uplus U_{1,1} \uplus \cdots \uplus U_{1,a_1}$ and $V_1 = V_{1,0} \uplus V_{1,1} \uplus \cdots \uplus V_{1,b_1}$ is the witness partition of $(0,1,\ldots,a_1)|(0,1,\ldots,b_1)$-biregularity of $G[U_1 \cup V_1]$. Since $\bar{K}_1 = (|U_{1,0}|, |U_{1,1}|, \ldots, |U_{1,a_1}|)$, $\bar{L}_1 = (|V_{1,0}|, |V_{1,1}|, \ldots, |V_{1,b_1}|)$, the subgraph $G[U_1 \cup V_1]$ has size $\bar{K}_1|\bar{L}_1$. The proofs of (2)–(4) are similar. This completes the proof of Lemma 7.2. ∎

Next, we will show Lemma 7.3 which deals with the if direction of Theorem 7.1.

LEMMA 7.3. *For all size vectors $\bar{K}_1 \in \mathbb{N}^{a_1+1}$, $\bar{K}_2 \in \mathbb{N}^{a_2+1}$, $\bar{L}_1 \in \mathbb{N}^{b_1+1}$, and $\bar{L}_2 \in \mathbb{N}^{b_2+1}$, if there are*
(1) *a $(0,1,\ldots,a_1)|(0,1,\ldots,b_1)$-biregular graph with size $\bar{K}_1|\bar{L}_1$;*
(2) *an $(a_1, a_1-1, \ldots, 0)|(0^{+p}, 1^{+p}, \ldots, b_2^{+p})$-biregular graph with size $\bar{K}_1|\bar{L}_2$;*
(3) *a $(0^{+p}, 1^{+p}, \ldots, a_2^{+p})|(b_1, b_1-1, \ldots, 0)$-biregular graph with size $\bar{K}_2|\bar{L}_1$;*
(4) *an $(a_2^{+p}, (a_2-1)^{+p}, \ldots, 0^{+p})|(b_2^{+p}, (b_2-1)^{+p}, \ldots, 0^{+p})$-biregular graph with size $\bar{K}_2|\bar{L}_2$,*

then there is an $A_0|B_0$-biregular graph with size $(M_1, M_2)|(N_1, N_2)$, where $M_1 = \|\bar{K}_1^T\|$, $M_2 = \|\bar{K}_2^T\|$, $N_1 = \|\bar{L}_1^T\|$, and $N_2 = \|\bar{L}_2^T\|$.

Proof. Let $\bar{K}_1 = (K_{1,0}, \ldots, K_{1,a_1}) \in \mathbb{N}^{a_1+1}$, $\bar{K}_2 = (K_{2,0}, \ldots, K_{2,a_2}) \in \mathbb{N}^{a_2+1}$, $\bar{L}_1 = (L_{1,0}, \ldots, L_{1,b_1}) \in \mathbb{N}^{b_1+1}$, $\bar{L}_2 = (L_{2,0}, \ldots, L_{2,b_2}) \in \mathbb{N}^{b_2+1}$. Let U_1, U_2, V_1, V_2 be pairwise disjoint sets of elements such that

$$|U_1| = \|\bar{K}_1^T\|, \qquad |U_2| = \|\bar{K}_2^T\|, \qquad |V_1| = \|\bar{L}_1^T\|, \qquad |V_2| = \|\bar{L}_2^T\|.$$

We partition U_1, U_2, V_1, V_2 as follows:

$$\begin{aligned}
U_1 &:= U_{1,0} \uplus U_{1,1} \uplus \cdots \uplus U_{1,a_1}, &\quad& \text{where } (|U_{1,0}|, |U_{1,1}|, \ldots, |U_{1,a_1}|) = \bar{K}_1,\\
U_2 &:= U_{2,0} \uplus U_{2,1} \uplus \cdots \uplus U_{2,a_2}, && \text{where } (|U_{2,0}|, |U_{2,1}|, \ldots, |U_{2,a_2}|) = \bar{K}_2,\\
V_1 &:= V_{1,0} \uplus V_{1,1} \uplus \cdots \uplus V_{1,b_1}, && \text{where } (|V_{1,0}|, |V_{1,1}|, \ldots, |V_{1,b_1}|) = \bar{L}_1,\\
V_2 &:= V_{2,0} \uplus U_{2,1} \uplus \cdots \uplus V_{2,b_2}, && \text{where } (|V_{2,0}|, |U_{2,1}|, \ldots, |V_{2,b_2}|) = \bar{L}_2.
\end{aligned}$$

Suppose we have biregular graphs H_1, H_2, H_3, H_4, as stated in the hypotheses (1)–(4):

- H_1 is a $(0,1,\ldots,a_1)|(0,1,\ldots,b_1)$-biregular graph with size $\bar{K}_1|\bar{L}_1$;
- H_2 is an $(a_1, a_1-1, \ldots, 0)|(0^{+p}, 1^{+p}, \ldots, b_2^{+p})$-biregular graph with size $\bar{K}_1|\bar{L}_2$;
- H_3 is a $(0^{+p}, 1^{+p}, \ldots, a_2^{+p})|(b_1, b_1-1 \ldots, 0)$-biregular graph with size $\bar{K}_2|\bar{L}_1$;
- H_4 is an $(a_2^{+p}, (a_2-1)^{+p} \ldots, 0^{+p})|(b_2^{+p}, (b_2-1)^{+p}, \ldots, 0^{+p})$-biregular graph with size $\bar{K}_2|\bar{L}_2$.

We will combine all these graphs H_1, H_2, H_3, H_4 into one $A_0|B_0$-biregular graph G with size $(M_1, M_2)|(N_1, N_2)$. See Figure 7 for an illustration. First, we make some observations.

- Note that H_1 is a $(0,1,\ldots,a_1)|(0,1,\ldots,b_1)$-biregular graph with size $\bar{K}_1|\bar{L}_1$, matching the sizes of U_1 and V_1. So we may assume that U_1 is the set of vertices on the left-hand side, V_1 is the set of vertices on the right-hand side. We can also assume that $U_1 = U_{1,0} \uplus U_{1,1} \uplus \cdots \uplus U_{1,a_1}$ and $V_1 = V_{1,0} \uplus V_{1,1} \uplus \cdots$

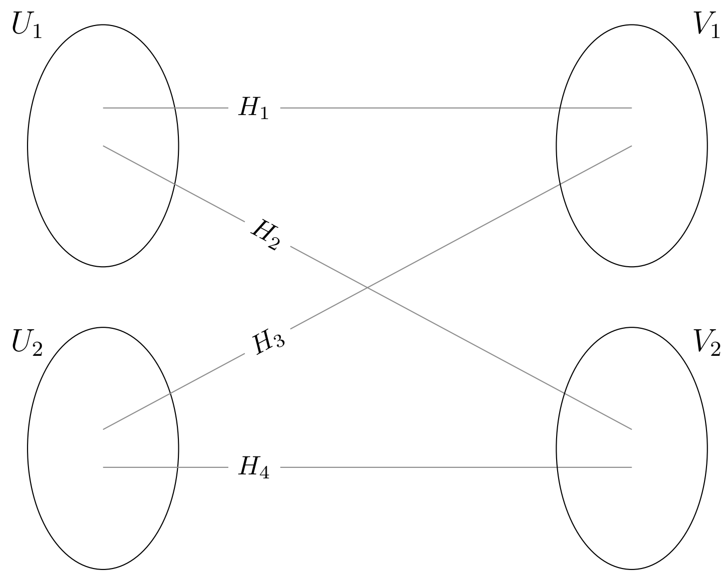

FIG. 7. *An illustration for the proof of Lemma* 7.3. *The graph H_1 contains only edges between the vertices in U_1 and V_1. The graph H_2 contains only edges between the vertices in U_1 and V_2. The graph H_3 contains only edges between the vertices in U_2 and V_1. The graph H_4 contains only edges between the vertices in U_2 and V_2. Thus, the sets of edges in H_1, H_2, H_3, H_4 are pairwise disjoint. The graph G obtained by combining all four of them is an $A_0|B_0$-biregular graph.*

$\uplus V_{1,b_1}$ is the witness partition for $(0, 1, \ldots, a_1)|(0, 1, \ldots, b_1)$-biregularity of H_1. Thus $H_1 = (U_1, V_1, R_1)$, where R_1 is the set of edges.

- In a similar manner, since H_2 is an $(a_1, a_1 - 1, \ldots, 0)|(0^{+p}, 1^{+p}, \ldots, b_2^{+p})$-biregular graph with size $\bar{K}_1|\bar{L}_2$, we may assume that U_1 is the set of vertices on the left-hand side, V_2 is the set of vertices on the right-hand side, and that $U_1 = U_{1,0} \uplus U_{1,1} \uplus \cdots \uplus U_{1,a_1}$ and $V_2 = V_{2,0} \uplus V_{2,1} \uplus \cdots \uplus V_{2,b_2}$ is the witness partition of $(a_1, a_1 - 1, \ldots, 0)|(0^{+p}, 1^{+p}, \ldots, b_2^{+p})$-biregularity of H_2.
  We can thus write $H_2 = (U_1, V_2, R_2)$, where R_2 is the set of edges. Note that R_1 and R_2 are disjoint since R_1 contains only edges between vertices in U_1 and vertices in V_1, whereas R_2 contains only edges between vertices in U_1 and vertices in V_2.
- Analogously to what we observed about H_2, since H_3 is a $(0^{+p}, 1^{+p}, \ldots, a_2^{+p})|(b_1, b_1 - 1, \ldots, 0)$-biregular graph with size $\bar{K}_2|\bar{L}_1$, we may assume that U_2 is the set of vertices on the left side, V_1 is the set of vertices on the right side, and that $U_2 = U_{2,0} \uplus U_{2,1} \uplus \cdots \uplus U_{2,a_2}$ and $V_1 = V_{1,0} \uplus V_{1,1} \uplus \cdots \uplus V_{1,b_1}$ is the witness partition of $(0^{+p}, 1^{+p}, \ldots, a_2^{+p})|(b_1, b_1 - 1, \ldots, 0)$-biregularity of H_3. We write $H_3 = (U_2, V_1, R_3)$, where R_3 is the set of edges and again note that R_1, R_2, R_3 are pairwise disjoint.
- Finally, since H_4 is an $(a_2^{+p}, (a_2 - 1)^{+p} \ldots, 0^{+p})|(b_2^{+p}, (b_2 - 1)^{+p}, \ldots, 0^{+p})$-biregular graph with size $\bar{K}_2|\bar{L}_2$, we may assume U_2 is the set of vertices on the left side, V_2 is the set of vertices on the right, and that $U_2 = U_{2,0} \uplus U_{2,1} \uplus \cdots \uplus U_{2,a_2}$ and $V_2 = V_{2,0} \uplus V_{2,1} \uplus \cdots \uplus V_{2,b_2}$ is the witness partition of $(a_2^{+p}, (a_2 - 1)^{+p}, \ldots, 0^{+p})|(b_2^{+p}, (b_2 - 1)^{+p}, \ldots, 0^{+p})$-biregularity of H_4.
  We can thus write $H_4 = (U_2, V_2, R_4)$, where R_4 is the set of edges and again note that R_1, R_2, R_3, R_4 are pairwise disjoint.

Let $G=(U_1\cup U_2, V_1\cup V_2, E)$, where $E=R_1\cup R_2\cup R_3\cup R_4$. That is, G is the graph union of all $H_1,\dots,H_4$. In fact, $G[U_1\cup V_1]$ is H_1, $G[U_1\cup V_2]$ is H_2, $G[U_2\cup V_1]$ is H_3, and $G[U_2\cup V_2]$ is H_4.

We will prove that G is an $A_0|B_0$-biregular graph with size $(M_1,M_2)|(N_1,N_2)$, where $M_1=|U_1|$, $M_2=|U_2|$, $N_1=|V_1|$, and $N_2=|V_2|$ by showing that

(1) every vertex in U_1 has degree a_1 and every vertex in U_2 has degree a_2^{+p}; and

(2) every vertex in V_1 has degree b_1 and every vertex in V_2 has degree b_2^{+p}.

To prove (1), note that:

- Since H_1 is a $(0,1,\dots,a_1)|(0,1,\dots,b_1)$-biregular graph, for every $j\in[0,a_1]$, every vertex $u\in U_{1,j}$ has degree j in H_1.
  Since H_2 is an $(a_1,a_1-1,\dots,0)|(0^{+p},1^{+p},\dots,b_2^{+p})$-biregular graph, for every $j\in[0,a_1]$, every vertex $u\in U_{1,j}$ has degree (a_1-j) in H_2.
  Therefore, for each $j\in[0,a_1]$, every vertex $u\in U_{1,j}$ has degree $j+(a_1-j)=a_1$ in the graph G.
- Similarly, since H_3 is a $(0^{+p},1^{+p},\dots,a_2^{+p})|(b_1,b_1-1,\dots,0)$-biregular graph, for every $j\in[0,a_2]$, every vertex $u\in U_{2,j}$ has degree j^{+p} in H_3.
  Since H_4 is an $(a_2^{+p},(a_2-1)^{+p}\dots,0^{+p})|(b_2^{+p},(b_2-1)^{+p},\dots,0^{+p})$-biregular graph, for every $j\in[0,a_2]$, every vertex $u\in U_{2,j}$ has degree $(a_2-j)^{+p}$ in H_4.
  Therefore, for every $j\in[0,a_2]$, each vertex $u\in U_{2,j}$ has degree $j^{+p}+(a_2-j)^{+p}=a_2^{+p}$ in the graph G.

The proof of (2) is similar. ∎

7.2. The general reduction from nonsimple to simple. We now give the general process which makes use of the idea above. In this section we will deal directly with complete biregular graphs. Let $A\in\mathbb{N}_{+p}^{t\times m}$ and $B\in\mathbb{N}_{+p}^{t\times n}$ be arbitrary degree matrices. We will show that every complete $A|B$-biregular graph can be decomposed into a collection of complete simple biregular graphs.

The idea is similar to the one in subsection 7.1. Let $G=(U,V,E_1,\dots,E_t)$ be a complete $A|B$-biregular graph. We let q be the maximal (finite) offset found in A and B. For each color $i\in[t]$, we call a vertex v an *E_i-neighbor* of a vertex u, if v is adjacent to u via an E_i-edge.

Suppose $U=U_1\uplus\cdots\uplus U_n$ and $V=V_1\uplus\cdots\uplus V_n$ is the witness partition of $A|B$-biregularity of G. For each $j\in[m]$, we further partition each U_j,

$$U_j \;=\; U_{j,g_1}\uplus\cdots\uplus U_{j,g_k},$$

where $g_1,\dots,g_k:[t]\times[n]\to\{0,1,\dots,q,0^{+p},1^{+p},\dots,q^{+p}\}$ are functions and for each color $i\in[t]$, for each $\ell\in[k]$, each vertex $u\in U_{j,g_\ell}$ has $g_\ell(i,1)$ E_i-neighbors in the set V_1, $g_\ell(i,2)$ E_i-neighbors in the set V_2, and so on to $g_\ell(i,n)$ E_i-neighbors in the set V_n. See Figure 8.[9] To ensure that each vertex in U_j has E_i-degree $A_{i,j}$ for every color $i\in[t]$, we require that $g_\ell(i,1)+\cdots+g_\ell(i,n)=A_{i,j}$. Note that if $A_{i,j}$ is a fixed entry, then all $g_\ell(i,1),\dots,g_\ell(i,n)$ are fixed entries. If $A_{i,j}$ is a periodic entry, then all $g_\ell(i,1),\dots,g_\ell(i,n)$ are periodic entries.

In the same way, for each $j'\in[n]$, we further partition each set $V_{j'}$,

$$V_{j'} \;=\; V_{j',h_1}\uplus\cdots\uplus V_{j',h_k},$$

where $h_1,\dots,h_k:[t]\times[m]\to\{0,1,\dots,q,0^{+p},1^{+p},\dots,q^{+p}\}$ are functions and for each color $i\in[t]$, for each $\ell\in[k]$, every vertex $u\in V_{j',h_\ell}$ has $h_\ell(i,1)$ E_i-neighbors in the set

[9]The partitioning of U_j into $U_{j,g_1}\uplus\cdots\uplus U_{j,g_k}$ is similar to how we partition the set $U_1=U_{1,0}\uplus\cdots\uplus U_{1,a_1}$ in Lemma 7.2 where for each $j\in[0,a_1]$, each vertex in $U_{1,j}$ has j neighbors in the set V_1 and (a_1-j) neighbors in the set V_2.

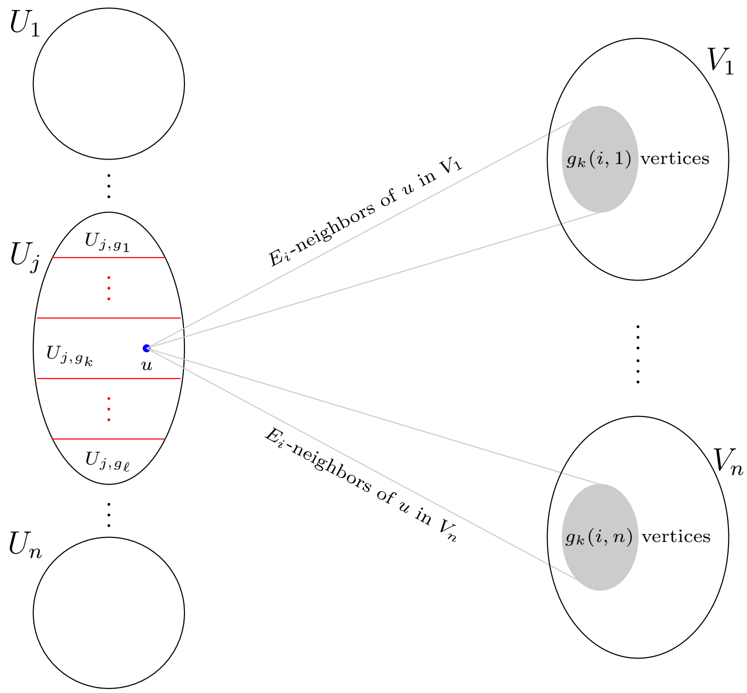

FIG. 8. *Suppose G is an $A|B$-biregular graph with $U = U_1 \uplus \cdots \uplus U_m$ and $V = V_1 \uplus \cdots \uplus V_m$ being the witness partition. We partition U_j according to the functions $g_1, \ldots, g_k : [t] \times [n] \to \{0, 1 \ldots, q, 0^{+p}, 1^{+p}, \ldots, q^{+p}\}$, where for each $\ell \in [k]$, each vertex in U_{j,g_ℓ} has $g_\ell(i, 1)$ E_i-neighbors in V_1, $g_\ell(i, 2)$ E_i-neighbors in V_2, and so on to $g_\ell(i, n)$ E_i-neighbors in V_n. Note: color appears only in the online article.*

U_1, $h_\ell(i, 2)$ E_i-neighbors in the set U_2, and so on to $h_\ell(i, m)$ E_i-neighbors in the set U_m.

We will show that every complete $A|B$-biregular graph G with witness partition $U = U_1 \uplus \cdots \uplus U_m$ and $V = V_1 \uplus \cdots \uplus V_n$ can be decomposed into complete simple biregular graphs in the sense that for each $j \in [m]$ and each $j' \in [n]$, the induced subgraph $G[U_j \cup V_{j'}]$ is a complete simple biregular graph with witness partition $U_j = U_{j,g_1} \uplus \cdots \uplus U_{j,g_k}$ and $V_{j'} = V_{j',h_1} \uplus \cdots \uplus V_{j',h_k}$. Such decomposition is also sufficient to capture all possible complete $A|B$-biregular graphs. We will formalize this idea in the next paragraphs.

We first need some terminology.

DEFINITION 7.4. *For each $j \in [m]$, we define* a behavior function of column j in A *to be a function $g : [t] \times [n] \to \{0, 1, \ldots, q, 0^{+p}, 1^{+p}, \ldots, q^{+p}\}$ such that*

- $A_{*,j} = \begin{pmatrix} g(1,1) + \cdots + g(1,n) \\ g(2,1) + \cdots + g(2,n) \\ \vdots \\ g(t,1) + \cdots + g(t,n) \end{pmatrix}$;
- *for each color $i \in [t]$, if $A_{i,j}$ is a fixed entry, then $g(i, 1), \ldots, g(i, n)$ are all fixed entries;*
- *for each color $i \in [t]$, if $A_{i,j}$ is a periodic entry, then $g(i, 1), \ldots, g(i, n)$ are all periodic entries.*

In a similar manner for each $j' \in [n]$, we define a behavior function of column j' in B *to be a function* $h : [t] \times [m] \to \{0, 1, \dots, q, 0^{+p}, 1^{+p}, \dots, q^{+p}\}$ *such that*

- $B_{*,j'} = \begin{pmatrix} h(1,1) + \cdots + h(1,m) \\ h(2,1) + \cdots + h(2,m) \\ \vdots \\ h(t,1) + \cdots + h(t,m) \end{pmatrix}$;
- *for each color $i \in [t]$, if $B_{i,j'}$ is a fixed entry, then $h(i,1), \dots, h(i,n)$ are all fixed entries;*
- *for each color $i \in [t]$, if $B_{i,j'}$ is a periodic entry, then $h(i,1), \dots, h(i,n)$ are all periodic entries.*

For each $j \in [m]$, let $g_{j,1}, \dots, g_{j,k}$ enumerate all behavior functions of column j in A. Similarly, for each $j' \in [n]$, let $h_{j',1}, \dots, h_{j',k}$ enumerate all behavior functions of column j' in B. Note that we assume that the number of behavior functions of column j in A is the same as the number of behavior functions of column j' in B for every $j \in [m]$ and every $j' \in [n]$. This is because we may "repeat" the same behavior function a few times in the enumeration $g_{j,1}, \dots, g_{j,k}$ and $h_{j',1}, \dots, h_{j',k}$.

For each $j \in [m]$, for each $j' \in [n]$, define the matrices $C_{j,j'}$ and $D_{j,j'}$:

$$C_{j,j'} := \begin{pmatrix} g_{j,1}(1,j') & g_{j,2}(1,j') & \cdots & g_{j,k}(1,j') \\ g_{j,1}(2,j') & g_{j,2}(2,j') & \cdots & g_{j,k}(2,j') \\ \vdots & \vdots & \ddots & \vdots \\ g_{j,1}(t,j') & g_{j,2}(t,j') & \cdots & g_{j,k}(t,j') \end{pmatrix}$$

and

$$D_{j,j'} := \begin{pmatrix} h_{j',1}(1,j) & h_{j',2}(1,j) & \cdots & h_{j',k}(1,j) \\ h_{j',1}(2,j) & h_{j',2}(2,j) & \cdots & h_{j',k}(2,j) \\ \vdots & \vdots & \ddots & \vdots \\ h_{j',1}(t,j) & h_{j',2}(t,j) & \cdots & h_{j',k}(t,j) \end{pmatrix}.$$

Note that for each color $i \in [t]$, if $A_{i,j}$ is a fixed entry, the values $g_{j,\ell}(i,1), \dots, g_{j,\ell}(i,n)$ are all fixed for each $\ell \in [k]$. Hence all the values $g_{j,1}(i,j'), \dots, g_{j,k}(i,j')$ are fixed, i.e., row i in $C_{j,j'}$ contains only fixed entries. Similarly, if $A_{i,j}$ is a periodic entry, the values $g_{j,\ell}(i,1), \dots, g_{j,\ell}(i,n)$ are all periodic for every $\ell \in [k]$. Hence all the values $g_{j,1}(i,j'), \dots, g_{j,k}(i,j')$ are periodic, i.e., row i in $C_{j,j'}$ contains only periodic entries. Therefore for each $j \in [m]$ and every $j' \in [n]$ $C_{j,j'}$ is a simple matrix. In a similar manner, we can argue that each $D_{j,j'}$ is a simple matrix.

We will show that every complete $A|B$-biregular graph can be decomposed into complete $C_{j,j'}|D_{j,j'}$-biregular graphs for every $j \in [m]$ and every $j' \in [n]$, as stated formally in Lemma 7.5.

LEMMA 7.5. *For every pair of size vectors $\bar{M} \in \mathbb{N}^m$ and $\bar{N} \in \mathbb{N}^n$, the statements* (a) *and* (b) *are equivalent.*

(a) *There is a complete $A|B$-biregular graph with size $\bar{M}|\bar{N}$.*

(b) *There are size vectors $\bar{K}_1, \dots, \bar{K}_m, \bar{L}_1, \dots, \bar{L}_n \in \mathbb{N}^k$ such that*

$$\bar{M} = (\|\bar{K}_1^T\|, \dots, \|\bar{K}_m^T\|) \qquad \text{and} \qquad \bar{N} = (\|\bar{L}_1^T\|, \dots, \|\bar{L}_n^T\|)$$

and for every $j \in [m]$ and for every $j' \in [n]$, there is a complete $C_{j,j'}|D_{j,j'}$-biregular graph with size $\bar{K}_j|\bar{L}_{j'}$.

The proof is a routine adaptation of Lemma 7.1, hence we omit the details. We describe here the main intuition. For (a) implies (b), suppose $G = (U, V, E_1, \ldots, E_t)$ is a complete $A|B$-biregular graph with size $\bar{M}|\bar{N}$. Let $U = U_1 \uplus \cdots \uplus U_m$ and $V = V_1 \uplus \cdots \uplus V_n$ be the witness partition. For every $j \in [m]$, for every $j' \in [n]$, we can show that each induced subgraph $G[U_j \cup V_{j'}]$ is a complete $C_{j,j'}|D_{j,j'}$-biregular graph with witness partition $U_j = U_{j,g_1} \uplus \cdots \uplus U_{j,g_k}$ and $V_{j'} = V_{j',h_1} \uplus \cdots \uplus V_{j',h_k}$, where $\bar{K}_j = (|U_{j,g_1}|, \ldots, |U_{j,g_k}|)$ and $\bar{L}_{j'} = (|V_{j',h_1}|, \ldots, |V_{j',h_k}|)$.

Conversely, for (b) implies (a), let $\bar{K}_1, \ldots, \bar{K}_m, \bar{L}_1, \ldots, \bar{L}_n \in \mathbb{N}^k$ be such that

$$\bar{M} = (\|\bar{K}_1^T\|, \ldots, \|\bar{K}_m^T\|) \qquad \text{and} \qquad \bar{N} = (\|\bar{L}_1^T\|, \ldots, \|\bar{L}_n^T\|).$$

Suppose for every $j \in [m]$ and for every $j' \in [n]$, there is a complete $C_{j,j'}|D_{j,j'}$-biregular graph $G_{j,j'}$ with size $\bar{K}_j|\bar{L}_{j'}$. Due to the matching sizes, we can assume that the set of vertices on the left-hand side of $G_{j,j'}$ is U_j and the set of vertices on the right-hand side of $G_{j,j'}$ is $V_{j'}$, where $|U_j| = \|\bar{K}_j^T\|$ and $|V_{j'}| = \|\bar{L}_{j'}^T\|$. Taking the disjoint union of all the graphs $G_{1,1} \cup \cdots \cup G_{m,n}$, we obtain a complete $A|B$-biregular graph G with size $\bar{M}|\bar{N}$.

Using Lemma 7.5, we can now define the formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{x}, \bar{y})$ as required in Theorem 3.2. We first explain the variables of the formula.

- For every $j \in [m]$, for every behavior function g of column j in A, we have a variable $X_{j,g}$. Let $\bar{X}_j = (X_{j,g_1}, \ldots, X_{j,g_k})$, where $g_1, \ldots, g_k$ are all the behavior functions of column j in A.
- Similarly, for every $j' \in [n]$, for every behavior function h of column j' in B, we have a variable $Y_{j',h}$. Let $\bar{Y}_{j'} = (Y_{j',h_1}, \ldots, Y_{j',h_k})$, where $h_1, \ldots, h_k$ are all the behavior functions of column j' in B.

Consider the formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{x}, \bar{y})$:

$$\exists \bar{X}_1 \cdots \exists \bar{X}_m \; \exists \bar{Y}_1 \cdots \exists \bar{Y}_n \qquad \bar{x} = (\|\bar{X}_1^T\|, \ldots, \|\bar{X}_m^T\|) \;\wedge\; \bar{y} = (\|\bar{Y}_1^T\|, \ldots, \|\bar{Y}_n^T\|) \tag{7.1}$$

$$\wedge \bigwedge_{j \in [m]} \bigwedge_{j' \in [n]} \mathsf{c\text{-}bireg}_{C_{j,j'}|D_{j,j'}}(\bar{X}_j, \bar{Y}_{j'}). \tag{7.2}$$

Note that $C_{j,j'}$ and $D_{j,j'}$ are simple matrices and the formula $\mathsf{c\text{-}bireg}_{C_{j,j'}|D_{j,j'}}(\bar{X}_j, \bar{Y}_{j'})$ is as defined in (6.4).

We show that the formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{x}, \bar{y})$ is correct, i.e., it captures all possible sizes of complete $A|B$-biregular graphs, as stated formally in Theorem 7.6.

THEOREM 7.6. *For every pair of degree matrices A and B, for every pair of size vectors $\bar{M}$ and $\bar{N}$, the formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{M}, \bar{N})$ holds in $\mathcal{N}$ if and only if there is a complete $A|B$-biregular graph with size $\bar{M}|\bar{N}$.*

The proof follows directly from Lemmas 7.5 and 6.3.

7.3. Proof of Theorem 3.3: Construction of the Presburger formula for complete regular digraphs. In section 7.2 we showed that given arbitrary degree matrices A and B, we can construct a Presburger formula that captures precisely the sizes of complete $A|B$-biregular graphs. The construction proceeds by reducing A and B into a collection of simple matrices. The proof for the digraph case is very similar to the biregular case. As in the 1-color case from subsection 4.2, the existence of $A|B$-regular digraphs with size $\bar{M}$ can be reduced to the existence of $A|B$-biregular graphs with size $\bar{M}|\bar{M}$. Indeed, an $A|B$-regular digraph G with size $\bar{M}$ can be encoded as an $A|B$-biregular graph G' with size $\bar{M}|\bar{M}$ by splitting each vertex w in G into two vertices u and v in G', where u is adjacent to all the outgoing edges and v to all the incoming edges. Thus, G' is a bipartite graph where the vertices on the left-hand side in G' are all the vertices with the outgoing edges and the vertices on the right-hand side are all the vertices with the incoming edges; see Figure 5 for an illustration.

The construction of the desired formula $\mathsf{c\text{-}reg}_{A|B}(\bar{x})$ that captures all possible sizes of a complete $A|B$-regular digraph can be done similarly to the one for complete biregular graphs. First, we construct a formula $\mathsf{c\text{-}reg}_{A|B}(\bar{x})$ when A and B are simple matrices, which is similar to section 6. The reduction from nonsimple matrices to simple matrices is similar to the one in section 7.2. We omit the details, since they are just a routine adaptation of the ones in sections 6 and 7.2.

8. Complexity of the decision procedures. We now analyze the complexity for each of the problems studied earlier. We begin with the biregular graph problems. We will then turn to the combined complexity of the decision procedure for the logic. Finally, we consider the complexity of the decision procedure for the logic when we fix a formula and vary its conjunction with a collection of ground facts—data complexity.

8.1. Complexity of the graph analysis. In this section we state the refined versions of the main results concerning biregular graph and digraph problems, now with complexity upper bounds. We do not have nontrivial lower bounds for these problems. As before, we only deal with the finite satisfiability. The analysis of general satisfiability can be found in the appendix.

LEMMA 8.1. *There is a nondeterministic Turing machine $\mathcal{M}$ that does the following: on input degree matrices $A \in \mathbb{N}_{+p}^{t\times m}$ and $B \in \mathbb{N}_{+p}^{t\times n}$, on every run r of $\mathcal{M}$, it outputs an existential Presburger formula $\varphi_r(\bar{x},\bar{y})$ such that*

- *each $\varphi_r(\bar{x},\bar{y})$ is of the form $\exists \bar{z}\ \widetilde{\varphi}_r(\bar{x},\bar{y},\bar{z})$, where each $\widetilde{\varphi}_r(\bar{x},\bar{y},\bar{z})$ is a conjunction of $O(mnt^4\delta(A,B)^4)$ linear (in)equations; and*
- *for every $(\bar{M},\bar{N}) \in \mathbb{N}^m \times \mathbb{N}^n$, there is a complete $A|B$-biregular graph with size $\bar{M}|\bar{N}$ if and only if there is a run r of $\mathcal{M}$ such that $\varphi_r(\bar{M},\bar{N})$ holds in $\mathcal{N}$.*

Moreover, $\mathcal{M}$ runs in time exponential in the size of A and B, where the coefficients of the input degree matrices and the output formula φ_r are in binary.

Proof. For arbitrary degree matrices $A \in \mathbb{N}_{+p}^{t\times m}$ and $B \in \mathbb{N}_{+p}^{t\times n}$, recall the formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{x},\bar{y})$ defined in (7.1),

$$\begin{aligned}&\exists \bar{X}_1 \cdots \exists \bar{X}_m\ \exists \bar{Y}_1 \cdots \exists \bar{Y}_n \qquad \bar{x} = (\|\bar{X}_1^T\|,\ldots,\|\bar{X}_m^T\|)\ \wedge\ \bar{y} = (\|\bar{Y}_1^T\|,\ldots,\|\bar{Y}_n^T\|)\\ &\quad \wedge \bigwedge_{j\in[m]} \bigwedge_{j'\in[n]} \mathsf{c\text{-}bireg}_{C_{j,j'}|D_{j,j'}}(\bar{X}_j,\bar{Y}_{j'}),\end{aligned}$$

where all $C_{j,j'}$ and $D_{j,j'}$ are simple matrices with t rows. Note that each variable in each $\bar{X}_j$ is of the form $X_{j,g}$, where $j \in [m]$ and $g : [t]\times[n] \to \{0,\ldots,q,0^{+p},\ldots,q^{+p}\}$ is a function and q is the maximal finite offset in A and B. Hence the number of bits to encode each $X_{j,g}$ is polynomial in the length of A and B. Similarly for each variable in each $\bar{Y}_{j'}$.

By Remark 6.4, each $\mathsf{c\text{-}bireg}_{C_{j,j'}|D_{j,j'}}(\bar{X}_j,\bar{Y}_{j'})$ is a disjunction of conjunctions of $O(t^4\delta(A,B)^4)$ (in)equations. Thus, the formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{x},\bar{y})$ is a disjunction of conjunctions of $O(mnt^4\delta(A,B)^4)$ (in)equations.

The desired Non-Deterministic Turing machine (NTM) $\mathcal{M}$ works as follows. On input A and B, it constructs the formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{x},\bar{y})$, where on each disjunction, it guesses which disjunct should hold. It outputs the constructed formula, which is a conjunction of $O(mnt^4\delta(A,B)^4)$ (in)equations and all the variables that are not in $\bar{x}$ and $\bar{y}$ are existentially quantified.

This by itself, of course, does not guarantee that the running time is only exponential, since the number of variables in the system may be more than exponential. Here we invoke results in [6, 12], which state that if a system of linear equations has a solution, it has a solution in which the number of variables taking nonzero values is bounded by a polynomial in the number of equations and in the length of the binary representation of the coefficients in the system.[10] Thus, when our algorithm constructs the formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{x},\bar{y})$, it also guesses the variables that take nonzero values, and ignores the remaining variables. Finally, applying Theorem 2.1, our decision procedure runs in (nondeterministic) exponential time. ∎

Lemma 8.2 is the directed graph analogue of Lemma 8.1, and the proof is similar.

LEMMA 8.2. *There is a nondeterministic Turing machine $\mathcal{M}$ that does the following: on input degree matrices $A \in \mathbb{N}_{+p}^{t\times m}$ and $B \in \mathbb{N}_{+p}^{t\times m}$, on every run r of $\mathcal{M}$, it outputs an existential Presburger formula $\varphi_r(\bar{x})$ such that*

- *each $\varphi_r(\bar{x})$ is of the form $\exists \bar{z}\ \widetilde{\varphi}_r(\bar{x},\bar{z})$, where each $\widetilde{\varphi}_r(\bar{x},\bar{z})$ is a conjunction of $O(m^2t^4\delta(A,B)^4)$ linear (in)equations; and*
- *for every $\bar{M} \in \mathbb{N}_\infty^m$, there is a complete $A|B$-regular digraph with size $\bar{M}$ if and only if there is a run r of $\mathcal{M}$ such that $\varphi_r(\bar{M})$ holds in $\mathcal{N}$.*

Moreover, $\mathcal{M}$ runs in time exponential in the size of A and B, where the coefficients of the input degree matrices and the output formula φ_r are in binary.

8.2. 2-NEXPTIME algorithm for the finite satisfiability of $\mathsf{FO}^2_{\mathbf{Pres}}$. We now give an analysis of the complexity of the decision procedure for our logic, based on the analysis of the complexity of the corresponding graph problems.

Recall that Π and $\mathcal{E}$ denote the set of 1- and 2-types, respectively. For finite satisfiability, a behavior function is a function $g:\{\text{out},\text{in}\}\times\mathcal{E}\times\Pi\to\mathbb{N}_{+p}$, where the codomain is $\{0,\dots,q,0^{+p},\dots,q^{+p}\}$ and q is the maximal offset in the u.p.s. S_i's. So, the total number of behavior functions is

$$m \ = \ (2q+2)^{2tn} \ = \ 2^{2tn\log(2q+2)},$$

where $t=|\mathcal{E}|$ and $n=|\Pi|$.

We enumerate all behavior functions $g_1,\dots,g_m$ and all 1-types $\pi_1,\dots,\pi_n$. The Presburger sentence PRES_ϕ is of the form

$$\mathsf{PRES}_\phi \ := \ \exists\bar{X}\ \mathsf{consistent}_1(\bar{X}) \ \wedge \ \mathsf{consistent}_2(\bar{X}) \wedge \left(\bigvee_{i\in[n],\ j\in[m]} X_{(\pi_i,g_j)} \neq 0 \right),$$

where $\bar{X}$ is a vector of variables $(X_{(\pi_1,g_1)},X_{(\pi_1,g_2)},\dots,X_{(\pi_n,g_m)})$.

The formula $\mathsf{consistent}_1(\bar{X})$ is

$$\mathsf{consistent}_1(\bar{X}) := \bigwedge_{\pi\text{ is incompatible},\ g\in\mathcal{G}} X_{\pi,g}=0 \quad\wedge \bigwedge_{(\pi,g)\in H} X_{\pi,g}=0 \\ \wedge \bigwedge_{g\text{ is not a good function},\ \pi\in\Pi} X_{\pi,g}=0,$$

[10]For example, Corollary 5 in [12] states that if a system $A\bar{x}=\bar{b}$ has a solution in $\mathcal{N}$, then it has a solution $\bar{x}$ such that the number of variables taking nonzero values is at most $2(d+1)(\log(d+1)+s+2)$, where d is the number of rows of A and s is the largest size of a coefficient in A and b (in binary representation).

where H is the set of all incompatible (π, g). Checking whether π and (π, g) are compatible/incompatible and whether g is a good function can be done in deterministic exponential time. So, this formula is negligible in our analysis.

Recall that for a 1-type π, $\bar{X}_\pi$ denotes the tuple of variables $(X_{\pi,g_1}, \dots, X_{\pi,g_m})$ The formula $\mathsf{consistent}_2$ is defined as

$$\mathsf{consistent}_2(\bar{X}) := \bigwedge_{1\leqslant i\leqslant n} \mathsf{c\text{-}reg}_{M^{\mathrm{out}}_{\pi_i}|M^{\mathrm{in}}_{\pi_i}}(\bar{X}_{\pi_i}) \ \wedge \bigwedge_{1\leqslant i<j\leqslant n} \mathsf{c\text{-}bireg}_{L_{\pi_j}|L^{\mathrm{rev}}_{\pi_i}}(\bar{X}_{\pi_i}, \bar{X}_{\pi_j}),$$

where

- $M^{\mathrm{out}}_{\pi_i}$ and $M^{\mathrm{in}}_{\pi_i}$ are matrices with size $t \times m$, and
- L_{π_j} and $L^{\mathrm{rev}}_{\pi_i}$ are matrices with size $2t \times m$.

Recall that t and m are the number of 2-types and behavior functions, respectively.

Using the Turing machine in Lemmas 8.1 and 8.2, the decision procedure can guess a formula $\mathsf{consistent}_2(\bar{x})$ where the total number of (in)equations is

$$O(n^2m^2t^4\delta(A,B)^4) \ = \ O(2^{4tn\log(2q+2)}n^2t^4\delta(A,B)^4), \tag{8.1}$$

where t and n are the numbers of 2-types and 1-types, respectively. That is, the number of (in)equations is doubly exponential in the size of the input formula.

The Turing machines in Lemmas 8.1 and 8.2 run in time exponential in the size of each $M^{\mathrm{out}}_{\pi_i}|M^{\mathrm{in}}_{\pi_i}$ and $L_{\pi_j}|L^{\mathrm{rev}}_{\pi_i}$ which, in turn, is exponential in the size of the input formula. So, altogether our decision procedure takes doubly exponential time to construct $\mathsf{consistent}_2(\bar{X})$. Applying Theorem 2.1, it runs in (nondeterministic) doubly exponential time.

Note that here we also invoke results in [6, 12]. Since the number of (in)equations in $\mathsf{consistent}_2(\bar{X})$ is only doubly exponential, if it has a solution, it has a solution in which the number of variables taking nonzero values is at most doubly exponential. Thus, the decision procedure also guesses the variables that take nonzero values, and ignores the remaining variables.

Thus, we have the 2-NEXPTIME upper bound for the finite satisfiability of $\mathsf{FO}^2_{\mathrm{Pres}}$, as stated formally as Theorem 8.3.

THEOREM 8.3. *The finite satisfiability of FO^2_{Pres} is in 2-NEXPTIME.*

8.3. 2-NEXPTIME algorithm for the general satisfiability of $\mathsf{FO}^2_{\mathbf{Pres}}$. In this subsection we will briefly explain that the same upper bound also holds for the general satisfiability of $\mathsf{FO}^2_{\mathrm{Pres}}$. First, we have the following lemma which is the analogue of Lemma 8.1 for the general case.

LEMMA 8.4. *There is a nondeterministic Turing machine $\mathcal{M}$ that does the following: on input of degree matrices $A \in \mathbb{N}^{t\times m}_{\infty,+p}$ and $B \in \mathbb{N}^{t\times n}_{\infty,+p}$, on every run r of $\mathcal{M}$, it outputs an existential Presburger formula $\varphi_r(\bar{x},\bar{y})$ such that*

- *each $\varphi_r(\bar{x},\bar{y})$ is of the form $\exists \bar{z}\ \widetilde{\varphi}_r(\bar{x},\bar{y},\bar{z})$, where each $\widetilde{\varphi}_r(\bar{x},\bar{y},\bar{z})$ is a conjunction of $O(mn2^tt^4\delta(A,B)^4)$ linear (in)equations; and*
- *for every $(\bar{M},\bar{N}) \in \mathbb{N}^m_\infty \times \mathbb{N}^n_\infty$, there is complete $A|B$-biregular graph with size $\bar{M}|\bar{N}$ if and only if there is a run r of $\mathcal{M}$ such that $\varphi_r(\bar{M},\bar{N})$ holds in $\mathcal{N}_\infty$.*

Moreover, $\mathcal{M}$ runs in time exponential in the size of A and B, where the coefficients of the input degree matrices and the output formula φ_r are in binary.

Note the additional factor 2^t in the number of linear (in)equations which is incurred in the construction of the formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{x},\bar{y})$ when A and B are simple matrices and may contain ∞ entries. The detailed analysis can be found in the appendix. The directed graph analogue is stated as Lemma 8.5.

LEMMA 8.5. *There is a nondeterministic Turing machine $\mathcal{M}$ that does the following: on input of degree matrices $A \in \mathbb{N}^{t\times m}_{\infty,+p}$ and $B \in \mathbb{N}^{t\times m}_{\infty,+p}$, on every run r of $\mathcal{M}$, it outputs an existential Presburger formula $\varphi_r(\bar{x})$ such that*

- *each $\varphi_r(\bar{x})$ is of the form $\exists \bar{z}\ \widetilde{\varphi}_r(\bar{x}, \bar{z})$, where each $\widetilde{\varphi}_r(\bar{x}, \bar{z})$ is a conjunction of $O(m^2 2^t t^4 \delta(A, B)^4)$ linear (in)equations; and*
- *for every $\bar{M} \in \mathbb{N}^m_\infty$, there is a complete $A|B$-regular digraph with size $\bar{M}$ if and only if there is a run r of $\mathcal{M}$ such that $\varphi_r(\bar{M})$ holds in $\mathcal{N}_\infty$.*

Moreover, $\mathcal{M}$ runs in time exponential in the size of A and B, where the coefficients of the input degree matrices and the output formula φ_r are in binary.

Another difference between the procedures for the finite and general satisfiability of $\mathsf{FO}^2_{\text{Pres}}$ is that the codomain of a behavior function for the general case is $\{\infty, 0, \ldots, q, 0^{+p}, \ldots, q^{+p}\}$, where q is the maximal (non-∞) offset in the u.p.s. S_i's. Then, the total number of behavior functions becomes

$$m \;=\; (2q+3)^{2tn} \;=\; 2^{2tn\log(2q+3)},$$

where t is the number of all 2-types and n is the number of all 1-types.

Similarly to the finite case, using the Turing machine in Lemmas 8.4 and 8.5, the decision procedure can guess a formula $\mathsf{consistent}_2(\bar{x})$ where the total number of (in)equations is

$$O(n^2 m^2 2^t t^4 \delta(A, B)^4) \;=\; O(2^{t+4tn\log(2q+3)} n^2 t^4 \delta(A, B)^4), \tag{8.2}$$

where t and n are the numbers of 2-types and 1-types, respectively. That is, the number of (in)equations is still doubly exponential in the size of the input formula. Using the algorithm in Theorem 2.1, the 2-NEXPTIME upper bound also holds for the general satisfiability of $\mathsf{FO}^2_{\text{Pres}}$, as stated formally as Theorem 8.6.

THEOREM 8.6. *The general satisfiability of $\mathsf{FO}^2_{\text{Pres}}$ is in 2-NEXPTIME.*

8.4. Data complexity of $\mathsf{FO}^2_{\text{Pres}}$ formulas. We now turn to families of formulas of the form $\phi \wedge \bigwedge_{D\in\mathcal{D}} D$, where ϕ is in the logic and the set $\mathcal{D}$ ranges over a finite collection of facts. We say that ϕ has NP *data complexity of (finite) satisfiability* if there is a nondeterministic algorithm that takes as input a finite set of ground atoms $\mathcal{D}$ and determines whether $\phi \wedge \bigwedge_{D\in\mathcal{D}} D$ is satisfiable, running in time polynomial in cardinality of $\mathcal{D}$.

Pratt-Hartmann [27] showed that C^2 formulas have NP data complexity of both satisfiability and finite satisfiability. Following the general approach to data complexity from [27], while plugging in our Presburger characterization of $\mathsf{FO}^2_{\text{Pres}}$, we can show that the same data complexity bound holds for $\mathsf{FO}^2_{\text{Pres}}$.

THEOREM 8.7. $\mathsf{FO}^2_{\text{Pres}}$ *formulas have* NP *data complexity of satisfiability and finite satisfiability.*

Proof. We give only the proof for finite satisfiability. We will follow closely the approach used for C^2 in section 4 of [27], and the terminology we use below comes from that work. We fix the $\mathsf{FO}^2_{\text{Pres}}$ sentence ϕ in the form 3.1.

Given a set of facts $\mathcal{D}$, our algorithm guesses a set of facts (including equalities) on elements of $\mathcal{D}$, giving us a finite set of facts $\mathcal{D}^+$ extending $\mathcal{D}$, but with the same domain as $\mathcal{D}$. We check that our guess is consistent with the universal part α and such that equality satisfies the usual transitivity and congruence rules.

Now consider 1-types and 2-types with an additional predicate Observable. Based on this extended language, we consider good functions as before, and define the formulas $\mathsf{consistent}_1$ and $\mathsf{consistent}_2$ based on them. 1-types that contain the predicate

Observable will be referred to as observable 1-types. The restriction of a behavior function to observable 1-types will be called an *observable behavior.* Given a structure M, an observable one-type π, and an observable behavior function g_0, we let M_{π,g_0} be the elements of M having 1-type π and observable behavior g_0, and we analogously let $\mathcal{D}_{\pi,g_0}$ be the elements of $\mathcal{D}$ whose 1-type and behavior in $\mathcal{D}^+$ match π and g_0.

We declare that all elements in $\mathcal{D}$ are in the predicate Observable. We add additional conjuncts to the formulas $\mathsf{consistent}_1$ and $\mathsf{consistent}_2$ stating that for each observable 1-type π and for each observable behavior function g_0, the total sum of the number of elements with 1-type π and a behavior function g extending g_0 (i.e., the cardinality of M_{π,g_0}) is the same as $|\mathcal{D}_{\pi,g_0}|$. Here the cardinality is being counted modulo equalities of $\mathcal{D}^+$.

At this point, our algorithm returns true exactly when the sentence obtained by existentially quantifying this extended set of conjuncts is satisfiable in the integers. The solving procedure is certainly in NP. In fact, since the number of variables is fixed, with only the constants varying, it is in PTIME [24].

We argue for correctness, focusing on the proof that when the algorithm returns true we have the desired model. Assuming the constraints above are satisfied, we get a graph, and from the graph we get a model M. M will clearly satisfy ϕ, but its domain does not contain the domain of $\mathcal{D}$. Letting O be the elements of M satisfying Observable, we know, from the additional constraints imposed, that the cardinality of O matches the cardinality of the domain of $\mathcal{D}$ modulo the equalities in $\mathcal{D}^+$, and for each observable 1-type π_o and observable behavior g_0, $|M_{\pi,g_0}| = |\mathcal{D}_{\pi,g_0}|$.

Fix an isomorphism λ taking each M_{π,g_0} to (equality classes of) $\mathcal{D}_{\pi,g_0}$. Create M' by redefining M on O by connecting pairs (o_1, o_2) via a 2-type μ exactly when $\lambda(o_1), \lambda(o_2)$ are connected via μ in $\mathcal{D}^+$. We can thus identify O with $\mathcal{D}^+$ modulo equalities in M'.

Clearly M' now satisfies the facts in $\mathcal{D}$. To see that M' satisfies ϕ, we simply note that since all of the observable behaviors are unchanged in moving from an element e in M to the corresponding element $\lambda(e)$ in M', and every such e modified has an observable type, it follows that the behavior of every element in M is unchanged in moving from M to M'. Since the 1-types are also unchanged, M' satisfies ϕ. $\square$

Note that the data complexity result here is best possible, since even for FO^2 the data complexity can be NP-hard [27].

9. The spectrum problem. As mentioned in the introduction, our Presburger definability result gives additional information about models of $\mathsf{FO}^2_{\mathrm{Pres}}$ sentences, allowing us to characterize the sets that can occur as cardinalities of models. Recall from the introduction that the spectrum of a sentence ϕ in any logic is the set of cardinalities of finite models of ϕ. We now use the prior tools to characterize the spectra for $\mathsf{FO}^2_{\mathrm{Pres}}$ sentences.

THEOREM 9.1. *From an $\mathsf{FO}^2_{\mathrm{Pres}}$ sentence ϕ, we can effectively construct a Presburger formula $\psi(n)$ such that $\mathcal{N} \models \psi(n)$ exactly when n is the size of a finite structure that satisfies ϕ, and similarly a formula $\psi_\infty(n)$ such that $\mathcal{N}_\infty \models \psi_\infty(n)$ exactly when n is the size of a finite or countably infinite model of ϕ.*

Proof. A *type/behavior profile* for a model $\mathcal{A}$ is the vector of cardinalities of the sets $A_{\pi,g}$, where π ranges of 1-types and g over behavior functions (for a fixed ϕ). Recall that in the proof of Theorem 3.6 we actually showed, in Lemmas 3.11 and 3.12, that we can construct existential Presburger formulas which define exactly the vectors of integers that arise as the type/behavior profiles of models of ϕ. The domain of the model can be broken up as a disjoint union of sets $A_{\pi,g}$, and thus its cardinality is a sum of numbers in this vector. We can thus add one additional integer variable x_{total} in PRES_ϕ, which will be free, with an additional equation stating that x_{total} is the sum of all $X_{\pi,g}$'s. This allows us to conclude definability of the spectrum. ◻

10. Related work. The biregular graph method was introduced and applied to C^2 in [19]. The case of 1-color is characterized by a Presburger formula that just expresses the equality of the number of edges calculated from either side of the bipartite graph. The nontrivial direction of correctness is shown via distributing edges and then merging. The case of fixed degree and multiple colors is done via an induction, using merging and then swapping to eliminate parallel edges. The case of unfixed degree is handled using a case analysis depending on whether sizes are big enough, but the approach is different from the one we apply here based on simple matrices followed by a reduction from nonsimple to simple.

Note that a more restricted version of the method is used to prove the decidability of FO^2 extended with two equivalence relations [18].

This work can be seen as a demonstration of the power of the biregular graph method to get new decidability results. We make heavy use of both techniques and results in [19], adapting them to the richer logic. The additional expressiveness of the logic requires the introduction of additional inductive arguments to handle the interaction of ordinary counting quantifiers and modulo counting quantification.

An alternative to the biregular graph method is the machinery developed by Pratt-Hartmann for analyzing the decidability and complexity of C^2 [25, 28], its fragments [26], and its extensions [29, 7]. It is clear that the approaches are closely related, despite the differing terminology. In [28] binary relationships that are tied to fixed numerical bounds are associated with "feature functions," while relationships that are not constrained realize "silent 2-types." At this point we cannot provide a more precise mapping, nor can we say whether it would be possible to extend the approach of [25] to our logic. An advantage of the biregular graph method is that it is transparent in how to extract more information about the shape of witness structures. While we imagine that results on spectra of formulas can be shown via either method, with an understanding of biregular graph problems related to a logic in hand, it is completely straightforward to draw conclusions about the spectrum. From an expository point of view, the biregular graph approach has the advantage that one deals with the combinatorics of the underlying problems with the logic abstracted away early on. But admittedly, the current arguments are complex in both approaches.

Characterizing the spectrum for general first-order formulas is quite a difficult problem, with ties to major open questions in complexity theory [11]. There are other logics, incomparable in expressiveness with $\mathsf{FO}^2_{\mathrm{Pres}}$, where periodicity of the spectrum has been proven [17]. The arguments have a different feel, since in these logics one can reduce to reasoning about forests.

The paper [4] shows decidability for a logic with incomparable expressiveness: the quantification allows a more powerful quantitative comparison, but must be *guarded*—restricting the counts only of sets of elements that are adjacent to a given element. Counting extensions of 1-variable logics are studied in [2].

11. Conclusion. We have shown the Presburger definability of the solution set to certain graph problems. Using this, we show that we can extend the powerful language two-variable logic with counting to include ultimately periodic counting quantifiers without sacrificing decidability, and without losing the effective definability of the spectrum of formulas within Presburger arithmetic.

A number of complexity questions are left open by our work. We have obtained a 2NEXPTIME bound on complexity of deciding satisfiability of the logic. However the only lower bound we know of is NEXPTIME, inherited from FO^2.

A natural question left open by our work is the connection with other extensions of two-variable logic with counting. It has been shown that two-variable logic with counting remains decidable in the presence of a linear order [8]. It has also been shown that decidability is maintained when one of the relations is restricted to be an equivalence relation [29]. One would like to know if there is a common decidable extension of our logic and one of (or, ideally, both of) these logics.

We also leave open a number of other complexity questions for biregular graph analysis problems. In particular, the line between PTIME and NP for the membership problem of subsection 3.1 (with cardinalities in unary) is open.

Appendix A. Scott normal form. In this appendix we prove that every $\mathsf{FO}^2_{\mathrm{Pres}}$ formula can be converted into the normal form used in the body of the paper,

$$\forall x \forall y\ \alpha(x,y)\ \wedge\ \bigwedge_{i=1}^{k} \forall x \exists^{S_i} y\ \beta_i(x,y) \wedge x \neq y,$$

where $\alpha(x,y)$ is a quantifier-free formula, each $\beta_i(x,y)$ is an atomic formula, and each S_i is a u.p.s. Moreover, the conversion preserves the satisfiability and the spectra of $\mathsf{FO}^2_{\mathrm{Pres}}$ sentences.

We will first give a couple of lemmas.

LEMMA A.1. *Let $S \subseteq \mathbb{N}_\infty$, where $0 \notin S$ and let q be a unary predicate. Let $\phi(x,y)$ be a formula with free variables x and y. The sentence Ψ_1 that is defined as*

$$\Psi_1 :=\ \forall x\ \big(q(x)\ \rightarrow\ \exists^S y\ \phi(x,y)\big)$$

is equivalent to the sentence Ψ_2 that is defined as

$$\Psi_2 :=\ \forall x\ \exists^{S \cup \{0\}} y\ \big(q(x) \wedge \phi(x,y)\big)\ \wedge\ \forall x\ \exists^{\mathbb{N}_\infty - \{0\}} y\ \big(q(x) \rightarrow \phi(x,y)\big).$$

Proof. It is worth noting that $q(x) \wedge \phi(x,y)$ is equivalent to $(q(x) \rightarrow \phi(x,y)) \wedge (\neg q(x) \rightarrow \bot)$.

Let $\mathcal{A}$ be a structure. For an element $a \in A$, define $W_{a,\phi(x,y)}$ as follows:

$$W_{a,\phi(x,y)} :=\ \{b \in A \mid (\mathcal{A}, x/a, y/b) \models \phi(x,y)\};$$

that is, $W_{a,\phi(x,y)}$ is the set of elements that can be assigned to y so that $\phi(x,y)$ holds, when x is assigned with element a.

Suppose $\mathcal{A} \models \Psi_1$. So, for every $a \in q^{\mathcal{A}}$, $|W_{a,\phi}| \in S$. Thus we have

$$\mathcal{A}, x/a \models \exists^S y\ q(x) \rightarrow \phi(x,y) \quad \text{and} \quad \mathcal{A}, x/a \models \exists^S y\ q(x) \wedge \phi(x,y). \tag{A.1}$$

For every $a \notin q^{\mathcal{A}}$, the following holds:

$$\mathcal{A}, x/a \models \exists^{|A|} y\ q(x) \rightarrow \phi(x,y) \quad \text{and} \quad \mathcal{A}, x/a \models \exists^{0} y\ q(x) \wedge \phi(x,y). \tag{A.2}$$

Combining (A.1) and (A.2), we have $\mathcal{A} \models \Psi_2$.

For the other direction, suppose $\mathcal{A} \models \Psi_2$. Since $\mathcal{A} \models \forall x \; \exists^{S\cup\{0\}} y \; \big(q(x) \wedge \phi(x,y)\big)$, for every $a \in A$, either $|W_{a,\phi(x,y)}| = 0$ or $|W_{a,\phi(x,y)}| \in S$. Since $\mathcal{A} \models \forall x \; \exists^{\mathbb{N}_\infty - \{0\}} y \; (q(x) \rightarrow \phi(x,y))$, the following holds, for every $a \in q^{\mathcal{A}}$:

$$|W_{a,\phi(x,y)}| \neq 0.$$

Thus, for every $a \in q^{\mathcal{A}}$, $|W_{a,\phi(x,y)}| \in S$. Therefore, $\mathcal{A} \models \Psi_1$. ∎

The next lemma is proven in a similar manner.

LEMMA A.2. *Let $S \subseteq \mathbb{N}_\infty$, where $0 \in S$ and let q be a unary predicate. Let $\phi(x,y)$ be a formula with free variables x and y. The sentence Ψ_1 defined as*

$$\Psi_1 := \; \forall x \; \big(q(x) \; \rightarrow \; \exists^S y \; \phi(x,y)\big)$$

is equivalent to the sentence Ψ_2 defined as

$$\Psi_2 := \; \forall x \; \exists^S y \; \big(q(x) \wedge \phi(x,y)\big).$$

Obviously, Lemmas A.1 and A.2 can be modified easily when $q(x)$ is any quantifier-free formula with free variable x.

Conversion into almost Scott normal form. We will first show how to convert an $\mathsf{FO}^2_{\mathrm{Pres}}$ sentence into an equisatisfiable sentence in almost Scott normal form:

$$\forall x \forall y \; \alpha(x,y) \;\; \wedge \;\; \bigwedge_{i=1}^{k} \forall x \exists^{S_i} y \; \beta_i(x,y). \tag{A.3}$$

That is, the requirement $x \neq y$ is dropped for $\beta_i(x,y)$ to hold. In fact, we get more than equisatisfiability: each model of our sentence can be expanded to a model of the normal form. This will be important for our result about the spectrum. In the remainder of this section we omit similar statements for brevity.

The conversion is a rather standard renaming technique from two-variable logic. Let Ψ be an $\mathsf{FO}^2_{\mathrm{Pres}}$ sentence. We first assume that Ψ does not contain any subformula of the form $\forall x \, \phi$, by rewriting thit em into the form $\exists^0 x \, \neg\phi$.

Whenever there is a subformula $\psi(x)$ in Ψ of the form $\exists^S y \; \phi(x,y)$, where $\phi(x,y)$ is quantifier free and S is a u.p.s., we perform a transformation. Let q be a fresh unary predicate, and replace the subformula $\psi(x)$ in Ψ with atomic $q(x)$, and add a sentence which states that $q(x)$ is equivalent to $\psi(x)$:

$$\forall x \; \big(q(x) \; \leftrightarrow \; \psi(x)\big)$$

which is equivalent to

$$\forall x \; \big(q(x) \; \rightarrow \; \exists^S y \; \phi(x,y)\big) \quad \wedge \quad \forall x \; \big(\neg q(x) \; \rightarrow \; \exists^{\mathbb{N}_\infty - S} y \; \phi(x,y)\big)$$

which, in turn, by Lemmas A.1 and A.2, can be converted into sentences of the form (A.3). We iterate this procedure until Ψ is in the almost Scott normal form described above.

Conversion into Scott normal form in (3.1). Now we provide the conversion from almost Scott normal form into Scott normal form. Note that

$$\forall x \exists^S y \; \beta(x,y)$$

is equivalent to

$$\forall x\big(\neg\beta(x,x) \ \rightarrow\ \exists^{S} y\ \beta(x,y)\wedge x\neq y\big) \quad \wedge \quad \forall x\big(\beta(x,x) \ \rightarrow\ \exists^{S-1} y\ \beta(x,y)\wedge x\neq y\big),$$

where $S-1$ denotes the set $\{i-1 \mid i\in S\}$.

Applying Lemmas A.1 and A.2, a sentence of form (A.3) can be converted into an equisatisfiable sentence of the form

$$\forall x\forall y\ \alpha(x,y) \quad \wedge \quad \bigwedge_{i=1}^{k} \forall x\exists^{S_i} y\ \beta_i(x,y)\wedge x\neq y,$$

where each $\beta_i(x,y)$ is quantifier free. To make it into Scott normal form, we introduce a new predicate $\gamma_i(x,y)$, for each $1\leqslant i\leqslant k$, and rewrite the sentence as follows:

$$\forall x\forall y\ \left(\alpha(x,y)\wedge\bigwedge_{i=1}^{k}\big(\gamma_i(x,y)\leftrightarrow\beta_i(x,y)\big)\right)\ \wedge\ \bigwedge_{i=1}^{k}\forall x\exists^{S_i} y\ \gamma_i(x,y)\wedge x\neq y.$$

The conversion described above takes $O(Cn)$ time, where n is the length of the original $\mathsf{FO}^2_{\mathrm{Pres}}$ sentence and the factor C is the complexity of computing the complement $\mathbb{N}_\infty - S$ of a u.p.s. S, which of course, depends on the representation of a u.p.s. However, we should note that the number of new atomic predicates introduced is linear in n.

Appendix B. The extension of section 4, the 1-color case, to handle infinite graphs. In this appendix we will extend the formulas in section 4 to handle all possible (finite and infinite) sizes of 1-color $A|B$-biregular graphs.

LEMMA B.1. *For every $A\in\mathbb{N}^{1\times m}_{\infty,+p}$ and $B\in\mathbb{N}^{1\times n}_{\infty,+p}$, there exists an (effectively computable) existential Presburger formula $\mathsf{bireg}_{A|B}(\bar{x},\bar{y})$ such that for every $(\bar{M},\bar{N})\in\mathbb{N}^m_\infty\times\mathbb{N}^n_\infty$, the formula $\mathsf{c}-\mathsf{bireg}_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}_\infty$ if and only if there is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$.*

B.1. Notation and terminology. We regard ∞ as a periodic entry, since ∞ is considered the same as ∞^{+p}. Intuitively, the reason is that when a vertex has degree ∞, adding p (or any arbitrary number) of additional new edges adjacent to it still make its degree ∞. A periodic entry which is not ∞ is called a *finite* periodic entry. We define offset(∞) to be ∞.

For degree vectors $\bar{a}$ and $\bar{b}$ that contain ∞ entries, we write $\delta(\bar{a},\bar{b})$ to denote the maximal finite entry in $(\mathrm{offset}(\bar{a}),\mathrm{offset}(\bar{b}),p)$. For example, if $\bar{a}=(3,\infty)$ and $\bar{b}=(2^{+5},4)$, then $\delta(\bar{a},\bar{b})$ is the maximal finite entry in $(3,\infty,2,4,5)$, which is 5. We let $\mathrm{per}(\bar{a})$ denote the set of indexes j, where a_j is a finite periodic entry and $\inf(\bar{a})$ to denote the sets of indexes j, where $a_j=\infty$. As before, $\mathrm{nz}(\bar{a})$ denotes the set of indexes j where $a_j\neq 0$.

We redefine the notion of big-enough when the degree vectors contain ∞ entries.

DEFINITION B.2. *Let $\bar{a}$ and $\bar{b}$ be degree vectors and let $\bar{M}$ and $\bar{N}$ be size vectors with the same length as $\bar{a}$ and $\bar{b}$, respectively. We say that $\bar{M}|\bar{N}$* is big-enough *for $\bar{a}|\bar{b}$, if each of the following holds:*
(a) $\max(\|\bar{M}^T\|_{\mathrm{nz}(\bar{a})},\|\bar{N}^T\|_{\mathrm{nz}(\bar{b})})\geqslant 2\delta(\bar{a},\bar{b})^2+1$,
(b) $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})}=0$ *or* $\geqslant\delta(\bar{a},\bar{b})^2+1$,
(c) $\|\bar{M}^T\|_{\inf(\bar{a})}=0$ *or* $\geqslant\delta(\bar{a},\bar{b})$,
(d) $\|\bar{N}^T\|_{\mathrm{per}(\bar{b})}=0$ *or* $\geqslant\delta(\bar{a},\bar{b})^2+1$,
(e) $\|\bar{N}^T\|_{\inf(\bar{b})}=0$ *or* $\geqslant\delta(\bar{a},\bar{b})$.

B.2. The formula for the case of big-enough sizes. We consider four scenarios for the sizes $\bar{M}|\bar{N}$ of $\bar{a}|\bar{b}$-biregular graphs:

(GS1) $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} = \|\bar{M}^T\|_{\mathrm{inf}(\bar{a})} = \|\bar{N}^T\|_{\mathrm{per}(\bar{b})} = \|\bar{N}^T\|_{\mathrm{inf}(\bar{b})} = 0$ (i.e., there are only vertices with fixed degree);

(GS2) $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} \neq 0$ and $\|\bar{M}^T\|_{\mathrm{inf}(\bar{a})} = \|\bar{N}^T\|_{\mathrm{per}(\bar{b})} = \|\bar{N}^T\|_{\mathrm{inf}(\bar{b})} = 0$ (i.e., there are vertices with finite periodic degrees on exactly one side, but no vertex with ∞ degree);

(GS3) $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})}, \|\bar{N}^T\|_{\mathrm{per}(\bar{b})} \neq 0$ and $\|\bar{M}^T\|_{\mathrm{inf}(\bar{a})} = \|\bar{N}^T\|_{\mathrm{inf}(\bar{b})} = 0$ (i.e., there are vertices with finite periodic degrees on both sides, but no vertex with ∞ degree),

(GS4) $\|\bar{M}^T\|_{\mathrm{inf}(\bar{a})} \neq 0$ or $\|\bar{N}^T\|_{\mathrm{inf}(\bar{b})} \neq 0$ (i.e., there are vertices with infinite degree).

The rest of this section is devoted to the formulas for each of the cases above. Scenarios (GS1)–(GS3) are similar to (S1)–(S3) in section 4. For completeness, we present the formulas for them, but without the correctness proofs. Scenario (GS4) is a new scenario that is not present in the finite biregular graph case.

The formula and argument for scenario (GS1). Consider the formula Gen-$\psi^1_{\bar{a}|\bar{b}}(\bar{x}, \bar{y})$:

$$\mathrm{offset}(\bar{a}) \cdot \bar{x} = \mathrm{offset}(\bar{b}) \cdot \bar{y} \ \wedge \ \|\bar{x}^T\|_{\mathrm{per}(\bar{a})} = \|\bar{x}^T\|_{\mathrm{inf}(\bar{a})} = \|\bar{y}^T\|_{\mathrm{per}(\bar{b})} = \|\bar{y}^T\|_{\mathrm{inf}(\bar{b})} = 0.$$

The last conjunct simply states that (GS1) holds.

LEMMA B.3. *For every pair of degree vectors $\bar{a}, \bar{b}$ and for every pair of size vectors $\bar{M}, \bar{N}$ such that $\bar{M}|\bar{N}$ is big-enough for $\bar{a}|\bar{b}$, the formula Gen-$\psi^1_{\bar{a}|\bar{b}}(\bar{M}, \bar{N})$ holds in $\mathcal{N}_\infty$ if and only if there is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$, where* (GS1) *holds.*

Proof. The proof is similar to Lemma 4.3. ∎

The formula and argument for scenario (GS2). Recall that (GS2) states that "there are vertices with finite periodic degrees on exactly one side, but no vertex with ∞ degree." By symmetry, we may assume that the vertices with finite periodic degrees are on the left. Consider the formula Gen-$\psi^2_{\bar{a}|\bar{b}}(\bar{x}, \bar{y})$:

$$\exists z \big(z \neq \infty \ \wedge \ \mathrm{offset}(\bar{a}) \cdot \bar{x} \ + \ pz \ = \ \mathrm{offset}(\bar{b}) \cdot \bar{y} \big)$$
$$\wedge \ \|\bar{x}^T\|_{\mathrm{per}(\bar{a})} \neq 0 \ \wedge \ \|\bar{x}^T\|_{\mathrm{inf}(\bar{a})} = \|\bar{y}^T\|_{\mathrm{per}(\bar{b})} = \|\bar{y}^T\|_{\mathrm{inf}(\bar{b})} = 0.$$

The last two conjuncts state that (GS2) holds.

LEMMA B.4. *For every pair of degree vectors $\bar{a}, \bar{b}$ and for every pair of size vectors $\bar{M}, \bar{N}$ such that $\bar{M}|\bar{N}$ is big-enough for $\bar{a}|\bar{b}$, the formula Gen-$\psi^2_{\bar{a}|\bar{b}}(\bar{M}, \bar{N})$ holds in $\mathcal{N}_\infty$ if and only if there is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$, where* (GS2) *holds.*

Proof. The proof is similar to Lemma 4.4. ∎

The formula and argument for scenario (GS3). Recall that (GS3) states that "there are vertices with finite periodic degrees on both sides, but no vertex with ∞ degree." Consider the formula Gen-$\psi^3_{\bar{a}|\bar{b}}(\bar{x}, \bar{y})$:

$$\exists z_1, z_2 \big(z_1 \neq \infty \ \wedge \ z_2 \neq \infty \ \wedge \ \mathrm{offset}(\bar{a}) \cdot \bar{x} + pz_1 = \mathrm{offset}(\bar{b}) \cdot \bar{y} + pz_2 \big)$$
$$\wedge \ \|\bar{x}^T\|_{\mathrm{per}(\bar{a})} \neq 0 \ \wedge \ \|\bar{y}^T\|_{\mathrm{per}(\bar{b})} \neq 0 \ \wedge \ \|\bar{x}^T\|_{\mathrm{inf}(\bar{a})} = \|\bar{y}^T\|_{\mathrm{inf}(\bar{b})} = 0.$$

LEMMA B.5. *For every pair of degree vectors $\bar{a}, \bar{b}$ and for every pair of size vectors $\bar{M}, \bar{N}$ such that $\bar{M}|\bar{N}$ is big-enough for $\bar{a}|\bar{b}$, the formula Gen-$\psi^3_{\bar{a}|\bar{b}}(\bar{M}, \bar{N})$ holds in $\mathcal{N}_\infty$ if and only if there is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$, where (GS3) holds.*

Proof. The proof is similar to Lemma 4.5. ∎

The formula and argument for scenario (GS4). Recall that (GS4) states that "there are vertices with infinite degree." Consider the formula Gen-$\psi^4_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$:

$$\text{(B.1)} \qquad \big(\|\bar{x}^T\|_{\inf(\bar{a})} \neq 0 \ \vee \ \|\bar{y}^T\|_{\inf(\bar{b})} \neq 0\big)$$
$$\text{(B.2)} \quad \wedge \ \big(\|\bar{x}^T\|_{\inf(\bar{a})} \neq 0 \ \rightarrow \ \|\bar{y}^T\|_{\mathrm{nz}(\bar{b})} = \infty\big) \ \wedge \ \big(\|\bar{y}^T\|_{\inf(\bar{b})} \neq 0 \ \rightarrow \ \|\bar{x}^T\|_{\mathrm{nz}(\bar{a})} = \infty\big).$$

Notice that, unlike the previous scenarios, this formula does not involve any edge counting on the finite entries. Instead, we will make use of the fact that infinite degree vertices give us a lot of flexibility in forming graphs that meet our specification.

LEMMA B.6. *For every pair of degree vectors $\bar{a},\bar{b}$ and for every pair of size vectors $\bar{M},\bar{N}$ such that $\bar{M}|\bar{N}$ is big-enough for $\bar{a}|\bar{b}$, the formula Gen-$\psi^4_{\bar{a}|\bar{b}}(\bar{M},\bar{N})$ holds in $\mathcal{N}_\infty$ if and only if there is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$, where* (GS4) *holds.*

Proof. Let $\bar{a},\bar{b}$ be degree vectors and $\bar{M}|\bar{N}$ be big-enough for $\bar{a}|\bar{b}$. For the if direction, suppose there is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$, where (GS4) holds. Thus, $\|\bar{M}^T\|_{\inf(\bar{a})} \neq 0$ or $\|\bar{N}^T\|_{\inf(\bar{a})} \neq 0$. If there is a vertex on the left with ∞ degree, there are infinitely many vertices with nonzero degree on the right. Symmetrically, if there is a vertex on the right with ∞ degree, there are infinitely many vertices with nonzero degree on the left. Therefore, Gen-$\psi^4_{\bar{a}|\bar{b}}(\bar{M},\bar{N})$ holds in $\mathcal{N}_\infty$.

We now prove the only if direction, assuming Gen-$\psi^4_{\bar{a}|\bar{b}}(\bar{M},\bar{N})$ holds in $\mathcal{N}_\infty$ and constructing an $\bar{a}|\bar{b}$-biregular graph $G=(U,V,E)$ with size $\bar{M}|\bar{N}$. Let m be the length of $\bar{a}$ and n be the length of $\bar{b}$. First, we pick pairwise disjoint sets $U_1,\dots,U_m$, where each $|U_j|=M_j$ and pairwise disjoint sets $V_1,\dots,V_n$, where each $|V_j|=N_j$. We define the set of vertices of our graph as $U=U_1\cup\dots\cup U_m$ and $V=V_1\cup\dots\cup V_n$.

We know $\|\bar{M}^T\|_{\inf(\bar{a})} \neq 0$ or $\|\bar{N}^T\|_{\inf(\bar{a})} \neq 0$. Hence we have at least one of $\|\bar{M}^T\|_{\inf(\bar{a})} \neq 0$ and $\|\bar{N}^T\|_{\mathrm{nz}(\bar{b})} = \infty$ or $\|\bar{N}^T\|_{\inf(\bar{b})} \neq 0$ and $\|\bar{M}^T\|_{\mathrm{nz}(\bar{a})} = \infty$.

We can break this down further into three cases:

(a) U is infinite and V is finite.

(b) U is finite and V is infinite.

(c) U is infinite and V is infinite.

Case (a)*:* We perform the following two steps.

- Step 1: Making the degrees of vertices in V correct.
  Let k be any index such that U_k is infinite. For every $j\in[n]$, for every vertex $v\in V_j$, we ensure that its degree is offset(b_j) by connecting v with some "nonadjacent" vertices from the set U_k—that is, vertices in U_k that are not yet adjacent to any vertices in V. Since U_k has an infinite supply of vertices, there are always such nonadjacent vertices for each vertex v. The purpose of picking nonadjacent vertices is that, after this step, every vertex in U has degree either 0 or 1.
- Step 2: Making the degrees of vertices in U correct.
  Let $V^\infty = \bigcup_{j\in\inf(\bar{b})} V_j$, i.e., the set of vertices in V that are supposed to have ∞ degree. Since $\|\bar{N}^T\|_{\inf(\bar{b})} \neq 0$, the set V^∞ is not empty. Moreover, since $\bar{M}|\bar{N}$ is big-enough, the cardinality $|V^\infty| \geqslant \delta(\bar{a},\bar{b})$.
  Note that the degree of every vertex in U is at most 1. For every $j\in[m]$, for every vertex $u\in U_j$, we ensure its degree is offset(b_j) by connecting u with some vertices in V^∞. This is possible since offset$(b_j)\leqslant\delta(\bar{a},\bar{b})$ for every $j\in[m]$.

Case (b)*:* is clearly symmetric to case (a).

Case (c)*:* We enumerate the elements $u_1, u_2, \ldots$ and $v_1, v_2, \ldots$ in U and V, respectively. We construct an $\bar{a}|\bar{b}$-biregular graph $G = (U, V, E)$ by iterating through all $\ell = 1, 2, \ldots$, where on each iteration ℓ, we do the following.

- We make the degree of u_ℓ "correct" in the sense that if j is the index where $u_\ell \in U_j$, we make its degree become $\mathrm{offset}(a_j)$.
- We make the degree of v_ℓ correct in the sense that if j is the index where $v_\ell \in U_j$, we make its degree become $\mathrm{offset}(b_j)$.

At the same time, while making the degrees of u_ℓ and v_ℓ correct, we ensure the following:

1. The degrees of the vertices $u_1, \ldots, u_{\ell-1}$ do not change and are already correct in the sense that for every $u \in \{u_1, \ldots, u_{\ell-1}\}$, if j is the index where $u \in U_j$, its degree is already $\mathrm{offset}(a_j)$.
2. The degrees of the vertices $v_1, \ldots, v_{\ell-1}$ do not change and are already correct in the sense that for every $v \in \{v_1, \ldots, v_{\ell-1}\}$, if j is the index where $v \in V_j$, its degree is already $\mathrm{offset}(b_j)$.
3. The degree of each vertex in $\{u_{\ell+1}, u_{\ell+2}, \ldots\} \cup \{v_{\ell+1}, v_{\ell+2}, \ldots\}$ is 0 or 1.
4. There are infinitely many vertices in $\{u_{\ell+1}, u_{\ell+2}, \ldots\}$ with degree 0.
5. There are infinitely many vertices in $\{v_{\ell+1}, v_{\ell+2}, \ldots\}$ with degree 0.

Since U (resp., V) is countable, every vertex $u \in U$ (resp., $v \in V$) has a finite index ℓ such that $u_\ell = u$ (resp., $v_\ell = v$). After the ℓth iteration the degree of u_ℓ (resp., v_ℓ) does not change any more. Thus, as the iteration index ℓ goes to ∞, the degree of every vertex is correct and we obtain an $\bar{a}|\bar{b}$-biregular graph G.

We now describe how to make the degree of u_ℓ correct. At the beginning of the ℓth iteration, the degree of u_ℓ is either 0 or 1. We make it correct by picking some zero degree vertices in $\{v_{\ell+1}, v_{\ell+2}\}$ and connecting them to u_ℓ. Such zero degree vertices exist and there are infinitely many of them. Of course, if the degree of u_ℓ is supposed to be 1, we do not need to pick any vertices. If the degree of u_ℓ is supposed to be ∞, we also make sure that there are still infinitely many zero degree vertices left in U. Observe also that the degrees of the vertices $u_1, \ldots, u_{\ell-1}, v_1, \ldots, v_{\ell-1}$ do not change. Making the degree of v_ℓ correct can be done symmetrically. ∎

As in subsection 4.1.2, to capture all possible sizes of $\bar{a}|\bar{b}$-biregular graphs there are only some fixed k cases to consider, where each case is either equal to or symmetric to one of the scenarios (GS1)–(GS4). We can enumerate all the formulas $\varphi_1(\bar{x}, \bar{y}), \ldots, \varphi_k(\bar{x}, \bar{y})$ that deal with each of the cases and define the formula Gen-$\psi_{\bar{a}|\bar{b}}(\bar{x}, \bar{y})$:

$$\bigvee_{i=1}^{k} \varphi_i(\bar{x}, \bar{y}). \tag{B.3}$$

Combining Lemmas B.3–B.6, Gen-$\psi_{\bar{a}|\bar{b}}(\bar{x}, \bar{y})$ captures precisely all the big-enough sizes $\bar{M}|\bar{N}$ of an $\bar{a}|\bar{b}$-biregular graph.

B.3. The formula for the case of not-big-enough sizes: Fixed size encoding. To capture the not-big-enough sizes, we use the same "fixed size encoding" technique as in subsection 4.1.3. Note that not-big-enough sizes mean that one of the conditions (a)–(e) is violated. So, either we have restricted the total size of the graphs (when condition (a) is violated) or at least one of the norms $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})}$, $\|\bar{M}^T\|_{\mathrm{inf}(\bar{a})}$, $\|\bar{N}^T\|_{\mathrm{per}(\bar{b})}$, $\|\bar{N}^T\|_{\mathrm{inf}(\bar{b})}$ is fixed to some number. Since we can deal with the first option by enumeration, we focus on the second. The idea is that we can use fixed size enumeration as in subsection 4.1.3, with additional minor extensions to handle vertices with ∞ degree. To illustrate, we will show the construction when the first two of the four norms above are fixed to some number, while the second two still satisfy the corresponding condition in big-enough. This corresponds to (b) and (c) being violated, while (a), (d), and (e) hold, in the definition of big-enough. In this case we will have vertices with periodic and infinite degrees on the left-hand side, but not too many.

Let $\bar{a}, \bar{b}$ be degree vectors. We will give the formula Gen-$\phi^{r_1,r_2}_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ to capture the sizes $\bar{M}|\bar{N}$ of all possible $\bar{a}|\bar{b}$-biregular graphs where each of the following holds.

- $\|\bar{M}^T\|_{\mathrm{nz}(\bar{a})} - r_1 - r_2 \geqslant 2\delta(\bar{a},\bar{b})^2 + 1$.
- $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} = r_1 \leqslant \delta(\bar{a},\bar{b})^2$.
- $\|\bar{M}^T\|_{\mathrm{inf}(\bar{a})} = r_2 \leqslant \delta(\bar{a},\bar{b}) - 1$.
- $\|\bar{N}^T\|_{\mathrm{per}(\bar{b})} = 0$ or $\geqslant \delta(\bar{a},\bar{b})^2 + 1$.
- $\|\bar{N}^T\|_{\mathrm{inf}(\bar{b})} = 0$ or $\geqslant \delta(\bar{a},\bar{b})$.

If the first bullet item does not hold, the number of edges is at most $3\delta(\bar{a},\bar{b})^2 + \delta(\bar{a},\bar{b})$, and the sizes of all these graphs can simply enumerated. The formula is defined inductively on $r_1 + r_2$ with the base case $r_1 + r_2 = 0$. Note that when $r_1 + r_2 = 0$, $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} = \|\bar{M}^T\|_{\mathrm{inf}(\bar{a})} = 0$, which means (b) and (c) are no longer violated.

For an integer $r_1, r_2 \geqslant 0$, we define the formula Gen-$\phi^{r_1,r_2}_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ as follows.

- When $r_1 = r_2 = 0$, Gen-$\phi^{r_1,r_2}_{\bar{a}|\bar{b}}(\bar{x},\bar{y})$ is defined as in Lemma 4.6.
- When $r_1 \geqslant 1$, let
$$\phi^{r_1-1,r_2}_{\bar{a}|\bar{b}}(\bar{x},\bar{y}) \ := \ \exists s \exists \bar{z}_0 \exists \bar{z}_1 \bigvee_{i \in \mathrm{per}(\bar{a})} \left(\begin{array}{l} x_i \neq 0 \ \wedge \ \bar{z}_0 + \bar{z}_1 = \bar{y} \\ \wedge \ s \neq \infty \\ \wedge \ \|\bar{z}_1^T\|_{\mathrm{nz}(\bar{b})} = \mathrm{offset}(a_i) + ps \\ \wedge \ \phi^{r_1-1,r_2}_{\bar{a}|(\bar{b},\bar{b}-\bar{1})}(\bar{x} - \mathbf{e}_i, \bar{z}_0, \bar{z}_1) \end{array} \right),$$
where $\mathbf{e}_i$ is the unit vector where the ith component is 1, and the lengths of $\bar{z}_0$ and $\bar{z}_1$ are the same as $\bar{y}$ The vector subtraction $\bar{b} - \bar{1}$ is defined as in subsection 4.1.3 extended with $\infty - 1 = \infty$.
- When $r_2 \geqslant 1$, let
$$\phi^{r_1,r_2-1}_{\bar{a}|\bar{b}}(\bar{x},\bar{y}) \ := \ \exists s \exists \bar{z}_0 \exists \bar{z}_1 \bigvee_{i \in \mathrm{inf}(\bar{a})} \left(\begin{array}{l} x_i \neq 0 \ \wedge \ \bar{z}_0 + \bar{z}_1 = \bar{y} \\ \wedge \ \|\bar{z}_1^T\|_{\mathrm{nz}(\bar{b})} = \infty \\ \wedge \ \phi^{r_1,r_2-1}_{\bar{a}|(\bar{b},\bar{b}-\bar{1})}(\bar{x} - \mathbf{e}_i, \bar{z}_0, \bar{z}_1) \end{array} \right),$$
where $\mathbf{e}_i$ is as in the previous case and the lengths of $\bar{z}_0$ and $\bar{z}_1$ are the same as $\bar{y}$. The vector subtraction $\bar{b} - \bar{1}$ is defined as in the previous case.

LEMMA B.7. *For every integer $r_1, r_2 \geqslant 0$, for every pair of degree vectors $\bar{a}, \bar{b}$, for every pair of size vectors $\bar{M}, \bar{N}$ such that*

- $\|\bar{M}^T\|_{\mathrm{nz}(\bar{a})} - r_1 - r_2 \geqslant 2\delta(\bar{a},\bar{b})^2 + 1$,
- $\|\bar{M}^T\|_{\mathrm{per}(\bar{a})} = r_1$,
- $\|\bar{M}^T\|_{\mathrm{inf}(\bar{a})} = r_2$,
- $\|\bar{N}^T\|_{\mathrm{per}(\bar{b})} = 0$ *or* $\geqslant \delta(\bar{a},\bar{b})^2 + 1$,
- $\|\bar{N}^T\|_{\mathrm{inf}(\bar{b})} = 0$ *or* $\geqslant \delta(\bar{a},\bar{b})$,

the formula $\phi^{r_1,r_2}_{\bar{a}|\bar{b}}(\bar{M},\bar{N})$ holds in $\mathcal{N}_\infty$ if and only if there is an $\bar{a}|\bar{b}$-biregular graph with size $\bar{M}|\bar{N}$.

Proof. The proof is by induction on $r_1 + r_2$ and is a routine adaptation of Lemma 4.7. ∎

As mentioned in subsection 4.1.3, the remaining not-big-enough cases can be captured by formulas similar to the one given above.

B.4. The proof in the 1-color case for regular digraphs. Recall that we define digraphs so that they have no self-loops. Similar to what was done in subsection 4.2, a digraph can be viewed as a bipartite graph by splitting every vertex u into two vertices, where one is adjacent to all the incoming edges, and the other to all the outgoing edges. Thus, $\bar{a}|\bar{b}$-biregular digraphs with size $\bar{M}$ can be characterized as $\bar{a}|\bar{b}$-biregular graphs with size $\bar{M}|\bar{M}$. The construction of the formula for all the sizes of $\bar{a}|\bar{b}$-regular digraphs can be done by a routine adaptation of the one in sections B.2 and B.3.

Appendix C. The extension of section 5 (simple multicolor graphs) to infinite graphs. We will extend the formulas in section 5 to accommodate all possible (finite and infinite) sizes of $A|B$-biregular graphs, where A and B are simple degree matrices, as stated formally in Lemma C.1.

LEMMA C.1. *For every pair of simple matrices $A \in \mathbb{N}_{\infty,+p}^{t\times m}$ and $B \in \mathbb{N}_{\infty,+p}^{t\times m}$, there exists an (effectively computable) existential Presburger formula $\mathsf{bireg}_{A|B}(\bar{x},\bar{y})$ such that for every pair of size vectors $\bar{M} \in \mathbb{N}_{\infty}^{m}$ and $\bar{N} \in \mathbb{N}_{\infty}^{n}$, the formula $\mathsf{bireg}_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}_{\infty}$ if and only if there is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$.*

This section is organized as follows. To deal with an ∞ entry, we need some new notation, introduced in section C.1. Section C.2 contains the construction of the formula for extra-big-enough sizes—a generalization of the ones in sections 5.2 and 5.3. Here there is a new case which is specific to an ∞ entry. We discuss the formula for the sizes that are not extra-big-enough—where no new ideas are needed—in section C.3.

C.1. Notation and terminology. Let A be a degree matrix with t rows and m columns. For nonempty subsets $R \subseteq [t]$, we write $A_{R,*}$ to denote the matrix obtained by keeping only the rows with indices in R, with no column being omitted. Likewise, for $J \subseteq [m]$, $A_{*,J}$ denotes the matrix obtained by keeping only the columns with indices in J, with no rows being omitted.

Recall that we regard an ∞ entry as a periodic entry. The *finite offset* of A, denoted by fin-offset(A) is the matrix obtained by replacing every ∞ entry in offset(A) with 0. That is, in fin-offset(A) we are concerned only with the non-∞ entries. For example, if $A = \left(\begin{smallmatrix} 2^{+p} & \infty \\ 0 & 3^{+p} \end{smallmatrix}\right)$, then offset$(A) = \left(\begin{smallmatrix} 2 & \infty \\ 0 & 3 \end{smallmatrix}\right)$ and fin-offset$(A) = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)$. Obviously, if A does not contain any ∞ entry, offset$(A) =$ fin-offset(A).

If A and B contain periodic or infinite entries, $\delta(A,B)$ denotes $\max(\|\text{fin-offset}(A)\|, \|\text{fin-offset}(B)\|, p)$. We also have to modify the notion of extra-big-enough in section 5.1 in order to take the ∞ entries into account.

DEFINITION C.2. *Let A and B be simple degree matrices with t rows. Let $\bar{M}$ and $\bar{N}$ be size vectors, where $\bar{M}|\bar{N}$ is appropriate for $A|B$. We say that $\bar{M}|\bar{N}$ is extra-big-enough for $A|B$, if for every $i \in [t]$*

(a) $\max(\|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})}, \|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})}) \geqslant 8t^2\delta(A,B)^4 + 1$,
(b) $\|\bar{M}^T\|_{\mathrm{per}(A_{i,*})} = 0$ *or* $\geqslant \delta(A,B)^2 + 1$,
(c) $\|\bar{M}^T\|_{\mathrm{inf}(A_{i,*})} = 0$ *or* $\geqslant t\delta(A,B)$,
(d) $\|\bar{N}^T\|_{\mathrm{per}(B_{i,*})} = 0$ *or* $\geqslant \delta(A,B)^2 + 1$,
(e) $\|\bar{N}^T\|_{\mathrm{inf}(B_{i,*})} = 0$ *or* $\geqslant t\delta(A,B)$.

C.2. Proof of Lemma C.1 for big-enough sizes. We divide the proof into three scenarios.

(GM1) $\|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})}, \|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})} \neq \infty$ for every $i \in [t]$ (i.e., the number of vertices with nonzero degree is finite, which means that the degree of every vertex must be finite).

(GM2) $\|\bar{M}^T\|_{\mathrm{inf}(A_{i,*})} = \|\bar{N}^T\|_{\mathrm{inf}(B_{i,*})} = 0$ for every $i \in [t]$ (i.e., the degree of every vertex is finite, but there may be infinitely many vertices).

(GM3) (the general case).

Note that (GM1) is strictly subsumed by (GM2), since in (GM2) the number of vertices with nonzero (finite) degree may be infinite. (GM2) is clearly strictly subsumed by (GM3). The rest of this section is devoted to the formulas for each of the scenarios above. The formula for scenario (GMi) will be used by the formula for cenario (GM$i+1$). Only (GM3), which deals with the possibility of vertices with infinite degree, will require substantial new work.

The formula and argument for scenario (GM1). For simple degree matrices A and B with t rows, consider the formula Gen-$\Psi^1_{A|B}(\bar{x}, \bar{y})$ given by

$$\begin{aligned}
\text{(C.1)} \qquad & \exists z_{1,1} \cdots \exists z_{1,t}\ \exists z_{2,1} \cdots \exists z_{2,t} \\
& \mathrm{offset}(A) \cdot \bar{x}^T + \begin{pmatrix} \alpha_1 p z_{1,1} \\ \vdots \\ \alpha_t p z_{1,t} \end{pmatrix} = \mathrm{offset}(B) \cdot \bar{y}^T + \begin{pmatrix} \beta_1 p z_{2,1} \\ \vdots \\ \beta_t p z_{2,t} \end{pmatrix} \\
& \wedge \bigwedge_{i \in [t]} z_{1,i} \neq \infty \ \wedge\ z_{2,i} \neq \infty \ \wedge\ \|\bar{x}^T\|_{\mathrm{nz}(A_{i,*})} \neq \infty \ \wedge\ \|\bar{y}^T\|_{\mathrm{nz}(B_{i,*})} \neq \infty \\
& \wedge \bigwedge_{i \in [t]} \|\bar{x}^T\|_{\mathrm{inf}(A_{i,*})} = \|\bar{y}^T\|_{\mathrm{inf}(B_{i,*})} = 0,
\end{aligned}$$

where $\alpha_i = 1$ if row i in A consists of periodic entries and is 0 otherwise,. Similarly $\beta_i = 1$ if row i in B consists of periodic entries and is 0 otherwise.

We claim that Gen-$\Psi^1_{A|B}(\bar{x}, \bar{y})$ captures all possible big-enough sizes $\bar{M}|\bar{N}$ of $A|B$-biregular graphs where (GM1) holds, as stated in Lemma C.3.

LEMMA C.3. *For each pair of simple degree matrices A and B, and each pair of size vectors $\bar{M}, \bar{N}$ such that $\bar{M}|\bar{N}$ is big-enough for $A|B$, the formula $\Psi^1_{A|B}(\bar{M}, \bar{N})$ holds in $\mathcal{N}_\infty$ if and only if there is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$, where* (GM1) *holds.*

Proof. This is similar to Lemma 5.9. ∎

The formula and argument for scenario (GM2). Recall that (GM2) states that there is no vertex with ∞ degree, but the number of edges may be infinite. The main idea for this scenario is to partition the edge colors into two kinds, depending on whether the number of edges is finite or infinite.

For simple matrices A and B with t rows, for a subset $R \subseteq [t]$, consider the formula Gen-$\Psi^{2,R}_{A|B}(\bar{x}, \bar{y})$ given by

$$\text{(C.2)} \qquad \text{Gen-}\Psi^1_{A_{R,*}|B_{R,*}}(\bar{x}, \bar{y}) \wedge \bigwedge_{i \in [t]} \|\bar{x}^T\|_{\mathrm{inf}(A_{i,*})} = \|\bar{y}^T\|_{\mathrm{inf}(B_{i,*})} = 0$$

$$\text{(C.3)} \qquad \wedge \bigwedge_{i \notin R} \|\bar{x}^T\|_{\mathrm{nz}(A_{i,*})} = \|\bar{y}^T\|_{\mathrm{nz}(B_{i,*})} = \infty,$$

where Gen-$\Psi^1_{A_{R,*}|B_{R,*}}(\bar{x}, \bar{y})$ is as defined in (C.1). Recall that $A_{R,*}$ is the matrix obtained from A by omitting all the rows not in R. When $R = [t]$, the formula Gen-$\Psi^{2,R}_{A|B}(\bar{x}, \bar{y})$ is the same as Gen-$\Psi^1_{A|B}(\bar{x}, \bar{y})$ defined for scenario (GM1).

Intuitively, Gen-$\Psi^{2,R}_{A|B}(\bar{x}, \bar{y})$ captures all the big-enough sizes of $A|B$-biregular graphs where (GM2) holds and $i \in R$ if and only if the number of E_i-edges is finite. This is stated formally as Lemma C.4.

LEMMA C.4. *For every pair of simple matrices A and B with t rows, for every $R \subseteq [t]$ and for every $\bar{M}|\bar{N}$ big-enough for $A|B$, the following are equivalent:*

1. *Gen-$\Psi^{2,R}_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}_\infty$.*
2. *There is an $A|B$-biregular graph $G=(U,V,E_1,\dots,E_t)$ with size $\bar{M}|\bar{N}$, where* (GM2) *holds and $R=\{i : |E_i| \neq \infty\}$.*

Proof. Let A and B be simple matrices with t rows and $R \subseteq [t]$. Let $\bar{M}|\bar{N}$ be big-enough for $A|B$.

We first prove "2 implies 1." Suppose $G=(U,V,E_1,\dots,E_t)$ is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$, where (GM2) holds and $R=\{i : |E_i| \neq \infty\}$. For every $i \in R$, since $|E_i| \neq \infty$, both $\|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})}$ and $\|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})}$ are not ∞. This means that G is an $A_{R,*}|B_{R,*}$-biregular graph, where (GM1) holds. By Lemma 5.9, Gen-$\Psi^2_{A_{R,*}|B_{R,*}}(\bar{M},\bar{N})$ holds.

Since (GM2) holds in G, there is no vertex with ∞ degree. Thus, we have

$$\bigwedge_{i\in[t]} \|\bar{M}^T\|_{\inf(A_{i,*})} = \|\bar{N}^T\|_{\inf(B_{i,*})} = 0.$$

Since every vertex has finite E_i-degree and $|E_i|=\infty$, for every $i \notin R$, the following conjunction holds:

$$\bigwedge_{i\notin R} \|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})} = \|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})} = \infty.$$

Combining all the assertions above, we see that Gen-$\Psi^{2,R}_{A|B}(\bar{M},\bar{N})$ holds.

Now we prove "1 implies 2." Suppose Gen-$\Psi^{2,R}_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}_\infty$. For simplicity, let $R=[\ell]$. Similarly to Lemma C.3, since $\|\bar{x}^T\|_{\inf(A_{i,*})} = \|\bar{y}^T\|_{\inf(B_{i,*})} = 0$ for every $i \in [t]$, we may assume that A and B do not contain an ∞ entry.

Since Gen-$\Psi^1_{A_{R,*}|B_{R,*}}(\bar{M},\bar{N})$ holds, by Lemma C.3 there is an $A_{R,*}|B_{R,*}$-biregular graph $G_0=(U,V,E_1,\dots,E_\ell)$ with size $\bar{M}|\bar{N}$, where (GM2) holds. Moreover, by (C.3), we have

$$\bigwedge_{i\notin R} \|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})} = \|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})} = \infty.$$

Hence for every $i \notin R$ we have

$$\mathrm{offset}(A_{i,*}) \cdot \bar{M} \;=\; \mathrm{offset}(B_{i,*}) \cdot \bar{N}.$$

By Lemma B.3, there is an $\mathrm{offset}(A_{i,*})|\mathrm{offset}(B_{i,*})$-biregular graph $G_i=(U,V,E_i)$ with size $\bar{M}|\bar{N}$ for every color $i \notin R$.

Consider the graph $G=(U,V,E_1,\dots,E_t)$ with size $\bar{M}|\bar{N}$. This graph G is almost $A|B$-biregular except that there may be an edge $(u,v) \in E_{i_1} \cap E_{i_2}$, for some $i_1 \neq i_2$. We use the edge swapping as in Lemma 5.6, to remove all such parallel edges.

Note that since G_0 is already $A_{R,*}|B_{R,*}$-biregular, at least one of i_1, i_2 is not in R. Suppose $i_1 \notin R$. Since $|E_{i_1}|=\infty$ and the degree of every vertex in G is finite, there is an E_{i_1}-edge (w,w') that is not incident to any of the neighbors of u and v. We can perform edge swapping (see also Figure 2) so that (u,v) is no longer an E_{i_1}-edge, without affecting the degree of each vertex. We perform edge swapping until there are no more parallel edges. This completes the proof of Lemma C.4. ∎

To wrap up scenario (GM2), we define formula Gen-$\Psi^2_{A|B}(\bar{x},\bar{y})$ for simple matrices A and B as

$$\bigvee_{R\subseteq[t]} \text{Gen-}\Psi^{2,R}_{A|B}(\bar{x},\bar{y}), \tag{C.4}$$

where each Gen-$\Psi^{2,R}_{A|B}(\bar{x},\bar{y})$ is defined in (C.2)–(C.3). This formula Gen-$\Psi^{2}_{A|B}(\bar{x},\bar{y})$ captures precisely all the extra-big-enough sizes of $A|B$-biregular graphs, where (GM2) holds, as stated formally as Lemma C.5.

LEMMA C.5. *For each pair of simple matrices A and B, and for each pair of size vectors $\bar{M},\bar{N}$ such that $\bar{M}|\bar{N}$ is big-enough for $A|B$, the formula Gen-$\Psi^{2}_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}_\infty$ if and only if there is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$, where* (GM2) *holds.*

Proof. Let A and B be simple matrices A and B with t rows and $\bar{M}|\bar{N}$ be big-enough for $A|B$.

We start with the if direction. Suppose there is an $A|B$-biregular graph $G = (U,V,E_1,\dots,E_t)$ with size $\bar{M}|\bar{N}$, where (GM2) holds. Let $R = \{i : |E_i| \neq \infty\}$. By Lemma C.4, Gen-$\Psi^{2,R}_{A|B}(\bar{M},\bar{N})$ holds. Thus, Gen-$\Psi^{2}_{A|B}(\bar{M},\bar{N})$ holds.

For the only if direction, suppose Gen-$\Psi^{2}_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}_\infty$. Let R be such that Gen-$\Psi^{2,R}_{A|B}(\bar{M},\bar{N})$ holds. By Lemma C.4, there is an $A|B$-biregular graph $G=(U,V,E_1,\dots,E_t)$ with size $\bar{M}|\bar{N}$, where (GM2) holds and $R=\{i : |E_i|\neq\infty\}$. ∎

The formula and argument for scenario (GM3). Recall that (GM3) is the general case where there may be vertices with infinite degree. The main idea here is to partition the edge colors E_i into two kinds, but this time depending on whether there are vertices with infinite E_i-degree. Let A and B be simple matrices with t rows and $R\subseteq[t]$. Consider the formula Gen-$\Psi^{3,R}_{A|B}(\bar{x},\bar{y})$ given by

$$\text{Gen-}\Psi^{2}_{A_{R,*}|B_{R,*}}(\bar{x},\bar{y}) \wedge \bigwedge_{i\notin R} \big(\|\bar{x}^T\|_{\inf(A_{i,*})}\neq 0 \ \vee\ \|\bar{y}^T\|_{\inf(B_{i,*})}\neq 0\big) \tag{C.5}$$

$$\wedge \bigwedge_{i\notin R} \|\bar{x}^T\|_{\inf(A_{i,*})}\neq 0 \ \rightarrow\ \|\bar{y}^T\|_{\mathrm{nz}(B_{i,*})}=\infty \tag{C.6}$$

$$\wedge \bigwedge_{i\notin R} \|\bar{y}^T\|_{\inf(B_{i,*})}\neq 0 \ \rightarrow\ \|\bar{x}^T\|_{\mathrm{nz}(A_{i,*})}=\infty, \tag{C.7}$$

where Gen-$\Psi^{2}_{A_{R,*}|B_{R,*}}(\bar{x},\bar{y})$ is as defined in (C.4). When $R=[t]$, Gen-$\Psi^{3,R}_{A|B}(\bar{x},\bar{y})$ is the same as Gen-$\Psi^{2}_{A|B}(\bar{x},\bar{y})$ defined for scenario (GM2).

Intuitively, Gen-$\Psi^{3,R}_{A|B}(\bar{x},\bar{y})$ captures all the big-enough sizes of $A|B$-biregular graphs, where R is the set of colors i such that every vertex has finite E_i-degree. We state this formally in Lemma C.6.

LEMMA C.6. *For every pair of simple matrices A and B with t rows, for every $R\subseteq[t]$, and for every pair of size vectors $\bar{M},\bar{N}$ such that $\bar{M}|\bar{N}$ is extra-big-enough for $A|B$, the following are equivalent:*

1. *Gen-$\Psi^{3,R}_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}_\infty$.*
2. *There is an $A|B$-biregular graph $G=(U,V,E_1,\dots,E_t)$ with size $\bar{M}|\bar{N}$, where $R=\{i : \text{every vertex in } G \text{ has finite } E_i\text{-degree}\}$.*

Proof. Let A and B be simple matrices with t rows and let $R\subseteq[t]$. Let $\bar{M}|\bar{N}$ be extra-big-enough for $A|B$.

We first prove 2 implies 1. Let G be an $A|B$-biregular graph with size $\bar{M}|\bar{N}$ and $R=\{i : \text{every vertex in } G \text{ has finite } E_i\text{-degree}\}$. Thus, G is also an $A_{R,*}|B_{R,*}$-biregular graph, where (GM2) holds. By Lemma C.5, Gen-$\Psi^{2}_{A_{R,*}|B_{R,*}}(\bar{M},\bar{N})$ holds.

By the definition of R, for every $i \notin R$, we have

$$\|\bar{M}^T\|_{\inf(A_{i,*})} \neq 0 \quad \text{or} \quad \|\bar{N}^T\|_{\inf(B_{i,*})} \neq 0.$$

If there is a vertex on one side with E_i-degree ∞, then there must be infinitely many vertices on the other side with nonzero E_i-degree. In other words, for every $i \notin R$

$$\|\bar{M}^T\|_{\inf(A_{i,*})} \neq 0 \ \rightarrow \ \|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})} = \infty, \text{ and}$$
$$\|\bar{N}^T\|_{\inf(B_{i,*})} \neq 0 \ \rightarrow \ \|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})} = \infty.$$

Therefore, the formula Gen-$\Psi^{3,R}_{A|B}(\bar{M}, \bar{N})$ holds.

We now prove 1 implies 2. Suppose Gen-$\Psi^{3,R}_{A|B}(\bar{M}, \bar{N})$ holds in $\mathcal{N}_\infty$. For simplicity, we may assume $R = [\ell]$. Since Gen-$\Psi^2_{A_{R,*}|B_{R,*}}(\bar{M}, \bar{N})$ holds, by Lemma C.5, there is an $A_{R,*}|B_{R,*}$-biregular graph $G_0 = (U, V, E_1, \ldots, E_\ell)$ with size $\bar{M}|\bar{N}$, where (GM2) holds. We will show how to extend G_0 to an $A|B$-biregular graph G with size $\bar{M}|\bar{N}$. Without loss of generality, we may assume that $R \neq [t]$. Otherwise, G_0 is already $A|B$-biregular and we are done.

In the following we fix $U = U_1 \uplus \cdots \uplus U_m$ and $V = V_1 \uplus \cdots \uplus V_n$ as the witness partition of $A_{R,*}|B_{R,*}$-biregularity of the graph G_0. We will add new edges to make G_0 into an $A|B$-biregular graph. In the following when we say "*we make the E_i-degree of a vertex $u \in U$ correct,*" we mean that we will add E_i-edges adjacent to u so that its E_i-degree becomes offset$(A_{i,j})$, where j is the index such that $u \in U_j$. Similarly for vertex $v \in V$.

We can break this down further into three cases—analogously to scenario (GS4):

(a) U is infinite and V is finite.

(b) U is finite and V is infinite.

(c) U is infinite and V is infinite.

Case (b) is symmetric to case (a), so we will only consider cases (a) and (c).

Case (a)*:* This case is a straightforward generalization of case (a) in (GS4). We perform the following two steps.

- Step 1: Making the E_i-degrees of vertices in V correct for every $i \notin R$.
  For each $i \notin R$, let k_i be any index such that U_{k_i} is infinite. For every $j \in [n]$, for every vertex $v \in V_j$, we ensure that its degree is offset$(B_{i,j})$ by connecting v with some nonadjacent vertices from the set U_{k_i}—that is, vertices in U_{k_i} that are not yet adjacent to any vertices in V. Since U_{k_i} has an infinite supply of vertices, there are always such nonadjacent vertices for each vertex v. The purpose of picking nonadjacent vertices is that, after this step, for every vertex $u \in U$ the sum $\sum_{i \notin R}$ (the E_i-degree of u) is either 0 or 1.
- Step 2: Making the degrees of vertices in U correct.
  For each $i \notin R$, let $V^{i,\infty} = \bigcup_{j \in \inf(B_{i,*})} V_j$, i.e., the set of vertices in V that are supposed to have ∞ E_i-degree. Since $\|\bar{N}^T\|_{\inf(B_{i,*})} \neq 0$, the set $V^{i,\infty}$ is not empty. Moreover, since $\bar{M}|\bar{N}$ is extra-big-enough, the cardinality $|V^{i,\infty}| \geqslant t\delta(A, B)$.
  Note that for each $i \notin R$, the sum $\sum_{i \notin R}$ (the E_i-degree of u) of every vertex $u \in U$ is at most 1. For every $j \in [m]$, for every vertex $u \in U_j$, we ensure its E_i-degree is offset$(B_{i,j})$, by connecting u with some vertices in $V^{i,\infty}$, for every $i \notin R$. This is possible since offset$(B_{i,j}) \leqslant \delta(A, B)$ for every $j \in [m]$.

Case (c)*:* For each $i \in [t]$, define the sets

$$U^{i,\mathrm{nz}} := \bigcup_{j \in \mathrm{nz}(A_{i,*})} U_j \quad \text{and} \quad V^{i,\mathrm{nz}} := \bigcup_{j \in \mathrm{nz}(B_{i,*})} V_j,$$
$$U^{i,\infty} := \bigcup_{j \in \inf(A_{i,*})} U_j \quad \text{and} \quad V^{i,\infty} := \bigcup_{j \in \inf(B_{i,*})} V_j.$$

Informally, $U^{i,\mathrm{nz}}$ and $V^{i,\mathrm{nz}}$ are the sets of vertices in U and V whose E_i-degree is supposed to be nonzero, while $U^{i,\infty}$ and $V^{i,\infty}$ are the vertices in U and V whose E_i-degree is supposed to be ∞.

Note that for each $i \notin R$, there are supposed to be vertices with infinite E_i-degree, which gives us a lot of flexibility in constructing the E_i-edges. We can use a technique similar to the one in scenario (GS4) from the previous appendix, which handled the case where some vertex has infinite degree in the single-color case. Note that for each $i \notin R$, we have one of the following:

(a) $U^{i,\mathrm{nz}}$ is infinite and $V^{i,\mathrm{nz}}$ is finite.

(b) $U^{i,\mathrm{nz}}$ is finite and $V^{i,\mathrm{nz}}$ is infinite.

(c) $U^{i,\mathrm{nz}}$ is infinite and $V^{i,\mathrm{nz}}$ is infinite.

That is, (a) holds for a subset of the colors, (b) holds for another subset, and (c) holds for the remaining colors. Constructing the E_i-edges by itself for each $i \notin R$ is comparatively easy, as shown in scenario (GS4). The main technical difficulty occurs when we try to make sure that the sets of constructed edges are still pairwise disjoint. Note also that here we do not have any guarantees about how big the partitions and degrees are in G_0. This limits us in using techniques such as edge swapping, which rely on having sufficiently many available edges.

In the following paragraphs, we will illustrate the new obstacle that arises. Suppose there are $i_1, i_2 \notin R$, where $i_1 \neq i_2$, such that

- $U^{i_1,\mathrm{nz}}$ is finite and $U^{i_2,\mathrm{nz}}$ is infinite;
- $V^{i_1,\mathrm{nz}}$ is infinite and $V^{i_2,\mathrm{nz}}$ is finite.

Since $i_1 \notin R$, the set $U^{i_1,\mathrm{nz}}$ contains vertices that are supposed to have infinite E_{i_1}-degree. Similarly, since $i_2 \notin R$, the set $V^{i_2,\mathrm{nz}}$ contain vertices that are supposed to have infinite E_{i_2}-degree. Assume, for convenience, that $U^{i_1,\mathrm{nz}} \subseteq U^{i_2,\mathrm{nz}}$ and $V^{i_2,\mathrm{nz}} \subseteq V^{i_1,\mathrm{nz}}$. See Figure 9 for an illustration.

If we construct the E_{i_1}-edges as in scenario (GS4), by connecting the vertices in $V^{i_2,\mathrm{nz}}$ with the vertices in $U^{i_1,\mathrm{nz}}$ with E_{i_1}-edges, there is a possibility that every vertex in $U^{i_1,\mathrm{nz}}$ is adjacent to every vertex in $V^{i_2,\mathrm{nz}}$ via E_{i_1}-edges. Thus, when we want to construct the E_{i_2}-edges, we can no longer connect the vertices in $U^{i_1,\mathrm{nz}}$ with the vertices in $V^{i_2,\mathrm{nz}}$ with E_{i_2}-edges, but the vertices in $V^{i_2,\mathrm{nz}}$ are the only vertices in G that are supposed to have nonzero E_{i_2}-degree. In other words, there is no more "room" to construct the E_{i_2}-edges. This issue will be circumvented by partitioning $U^{i_1,\infty} = X_0 \uplus X_1$ and $V^{i_2,\infty} = Y_0 \cup Y_1$ and constructing the E_{i_1}-edges so that

- vertices in X_0 are connected by E_{i_1}-edges only to vertices in Y_0;
- vertices in X_1 are connected by E_{i_1}-edges only to vertices in Y_1.

Then when we construct the E_{i_2}-edges, we will connect the vertices in X_0 with the vertices in Y_1 and the vertices in X_1 with the vertices in Y_0. The rest of the proof is devoted to the details of the construction.

Due to the technical difficulty described above, the following two sets of colors $F_{\text{nz-left}}, F_{\text{nz-right}} \subseteq [t]$ will need some special care:

$$F_{\text{nz-left}} := \{i \notin R : \|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})} \text{ is finite}\},$$
$$F_{\text{nz-right}} := \{i \notin R : \|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})} \text{ is finite}\}.$$

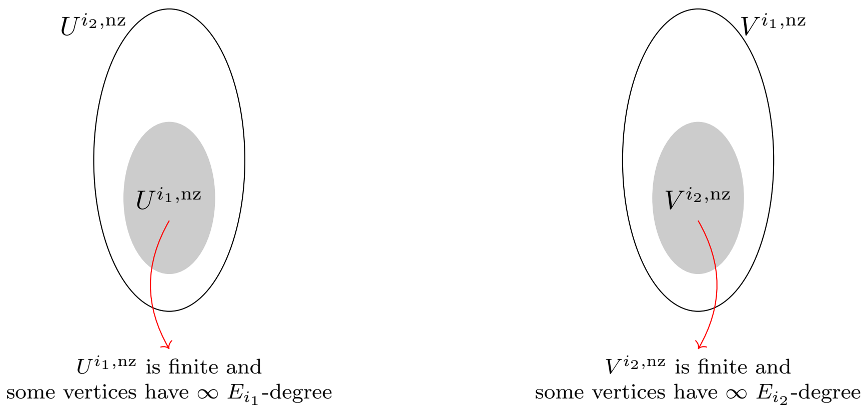

FIG. 9. *An illustration of the sets $U^{i_1,\mathrm{nz}}$, $U^{i_2,\mathrm{nz}}$, $V^{i_1,\mathrm{nz}}$, and $V^{i_2,\mathrm{nz}}$. When constructing the E_{i_1}-edges, we exploit the infinite E_{i_1}-degree vertices in $U^{i_1,\mathrm{nz}}$. Similarly, when constructing the E_{i_2}-edges, we exploit the infinite E_{i_2}-degree vertices in $V^{i_2,\mathrm{nz}}$. However, we have to make sure to avoid the possibility that every vertex in $U^{i_1,\mathrm{nz}}$ is already adjacent to every vertex in $V^{i_2,\mathrm{nz}}$ via E_{i_1}-edges, thus, leaving "no room" to connect them via E_{i_2}-edges. Note: color appears only in the online article.*

Intuitively, the set $F_{\text{nz-left}}$ is the set of color $i \notin R$ where there will only be finitely many vertices with non- zero E_i-degree on the *left*-hand side. The set $F_{\text{nz-right}}$ has the same intuitive meaning w.r.t to the vertices on the right-hand side.

We argue that $F_{\text{nz-left}}$ and $F_{\text{nz-right}}$ are disjoint. For every $i \notin R$, at least one of $\|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})}$ or $\|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})}$ is infinite. Moreover, since Gen-$\Psi^{3,R}_{A|B}(\bar{M},\bar{N})$ holds, we have

- $\|\bar{N}^T\|_{\inf(B_{i,*})} = 0$, $\|\bar{M}^T\|_{\inf(A_{i,*})} \neq 0$, and $\|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})} = \infty$, for every $i \in F_{\text{nz-left}}$.
  This is because for every $i \in F_{\text{nz-left}}$, $\|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})}$ is finite. Thus, the E_i-degree of every vertex in V must be finite, i.e., $\|\bar{N}^T\|_{\inf(B_{i,*})} = 0$. Since $i \notin R$, this means there are vertices on the left with ∞ E_i-degree, i.e., $\|\bar{M}^T\|_{\inf(A_{i,*})} \neq 0$. Therefore, the number of vertices on the right with non-zero E_i-degree must be infinite, i.e., $\|\bar{N}^T\|_{\mathrm{nz}(B_{i,*})} = \infty$;
- Similarly, $\|\bar{M}^T\|_{\inf(A_{i,*})} = 0$, $\|\bar{N}^T\|_{\inf(B_{i,*})} \neq 0$, and $\|\bar{M}^T\|_{\mathrm{nz}(A_{i,*})} = \infty$ for every $i \in F_{\text{nz-right}}$.

Therefore, $F_{\text{nz-left}}$ and $F_{\text{nz-right}}$ are disjoint.

Define the sets

$$U^{F_{\text{nz-left}},\infty} := \bigcup_{i \in F_{\text{nz-left}}} U^{i,\infty} \qquad \text{and} \qquad V^{F_{\text{nz-right}},\infty} := \bigcup_{i \in F_{\text{nz-right}}} V^{i,\infty}.$$

Note that for every $i \in F_{\text{nz-left}}$, the set $U^{i,\mathrm{nz}}$ is finite. Since $U^{i,\infty} \subseteq U^{i,\mathrm{nz}}$, the set $U^{i,\infty}$ is finite and hence so is the set $U^{F_{\text{nz-left}},\infty}$. By analogous reasoning, $V^{F_{\text{nz-right}},\infty}$ is also finite. Because $\bar{M}|\bar{N}$ is extra-big-enough for $A|B$, for every $i \in F_{\text{nz-left}}$, $|U^{i,\infty}| \geqslant t\delta(A,B)$ holds. Similarly, for every $i \in F_{\text{nz-right}}$, $|V^{i,\infty}| \geqslant t\delta(A,B)$.

The claim below is the formalization of the partition $X_0 \uplus X_1$ and $Y_0 \uplus Y_1$ described above.

CLAIM C.7. *Suppose $F_{\textit{nz-left}}, F_{\textit{nz-right}} \neq \emptyset$. Then*

- *there is a partition $X_0 \uplus X_1$ of $U^{F_{\textit{nz-left}},\infty}$ such that for every $i \in F_{\textit{nz-left}}$, both the sets $X_0 \cap U^{i,\infty}$ and $X_1 \cap U^{i,\infty}$ contain at least $\delta(A,B)$ vertices;*

- *there is a partition* $Y_0 \uplus Y_1$ *of* $V^{F_{\text{nz-right}},\infty}$ *such that for every* $i \in F_{\text{nz-right}}$*, both* $Y_0 \cap V^{i,\infty}$ *and* $Y_1 \cap V^{i,\infty}$ *contain at least* $\delta(A,B)$ *vertices.*

As explained above, the main difficulty in constructing the E_i-edges for color $i \in F_{\text{nz-left}}$ is that $|U^{i,\infty}|$ is finite but $|V^{i,\text{nz}}|$ is infinite, and for color $i \in F_{\text{nz-right}}$, $|V^{i,\infty}|$ is finite but $|U^{i,\text{nz}}|$ is infinite. The claim implies that there are sets X_0, X_1, Y_0, Y_1—each with enough vertices—allowing us to construct the E_i-edges for color $i \in F_{\text{nz-left}}$ as follows:

- To make every vertex in Y_0 have the correct E_i-degree, we connect it by E_i-edges only to the vertices in $X_0 \cap U^{i,\infty}$.
- To make every vertex in Y_1 have the correct E_i-degree, we connect it by E_i-edges only to the vertices in $X_1 \cap U^{i,\infty}$.

Note that for any color $i \in F_{\text{nz-left}}$, the set $U^{i,\text{nz}}$ is finite, hence the degree of each vertex in $V^{i,\text{nz}}$ must be finite, and bounded by $\delta(A,B)$. Since for every color $i \in F_{\text{nz-left}}$, the cardinalities of $X_0 \cap U^{i,\infty}$ and $X_1 \cap U^{i,\infty}$ are at least $\delta(A,B)$, there are "enough" vertices to connect vertices in Y_0 only with the vertices in $X_0 \cap U^{i,\infty}$ and vertices in Y_1 only with vertices in $X_1 \cap U^{i,\infty}$ via E_i-edges. After this construction, vertices in X_0 are not connected via E_i to any vertex in Y_1 for any color $i \in F_{\text{nz-left}}$. Likewise, vertices in X_1 are not connected via E_i to any vertex in Y_0, for any color $i \in F_{\text{nz-left}}$. This leaves some room for the construction of E_i-edges for each color $i' \in F_{\text{nz-right}}$ where we connect vertices in X_0 only to vertices in $Y_1 \cap V^{i',\infty}$ and vertices in X_1 only to vertices in $Y_0 \cap V^{i',\infty}$. See Figure 10 for an illustration.

Proof. (of Claim C.7) We prove the first item. The second one is similar. Initially, $X_0 = X_1 = \emptyset$. To achieve the desired property, we will add vertices to X_0 and X_1 by iterating on every $i \in F_{\text{nz-left}}$. On each iteration, we add at most $\delta(A,B)$ vertices to X_0 and X_1.

Suppose we are now iterating on some $i \in F_{\text{nz-left}}$. There are 4 cases:

- *Case* 1: $|X_0 \cap U^{i,\infty}| \geqslant \delta(A,B)$ and $|X_1 \cap U^{i,\infty}| \geqslant \delta(A,B)$. In this case, we do nothing and move on to the next $i \in F_{\text{nz-left}}$.
- *Case* 2: $|X_0 \cap U^{i,\infty}| < \delta(A,B)$ and $|X_1 \cap U^{i,\infty}| < \delta(A,B)$. Observe that $U^{i,\infty}$ contains $t\delta(A,B) \geqslant 2\delta(A,B)$ vertices. Thus, we can add some vertices from $U^{i,\infty}$ to X_0 and X_1 so that X_0 and X_1 are still disjoint and

$$|X_0 \cap U^{i,\infty}| \;=\; |X_1 \cap U^{i,\infty}| \;=\; \delta(A,B).$$

- *Case* 3: $|X_0 \cap U^{i,\infty}| < \delta(A,B)$ and $|X_1 \cap U^{i,\infty}| \geqslant \delta(A,B)$. Here, we see that

$$|U^{i,\infty}| \;\geqslant\; t\delta(A,B) \;>\; |F_{\text{nz-left}}|\delta(A,B) \;>\; (|F_{\text{nz-left}}|-1)\delta(A,B) \;\geqslant\; |X_1|.$$

  Thus, $U^{i,\infty}$ contains at least $\delta(A,B)$ vertices that are not yet in $X_0 \cup X_1$. We add some of these vertices into X_0 so that $|X_0 \cap U^{i,\infty}| = \delta(A,B)$.
- *Case* 4: $|X_0 \cap U^{i,\infty}| \geqslant \delta(A,B)$ and $|X_1 \cap U^{i,\infty}| < \delta(A,B)$. This case is symmetric to Case 3. ∎

Now we are ready to extend G_0 to an $A|B$-biregular graph G. Recall that G_0 is an $A_{R,*}|B_{R,*}$-biregular graph where $R \neq [t]$. By (C.6) and (C.7), at least one of U and V is infinite. We will show how to construct the E_i-edges in G for each $i \in [t]-R$. The construction will yield an $A|B$-biregular graph G with witness partition $U = U_1 \uplus \cdots \uplus U_m$ and $V = V_1 \uplus \cdots \uplus V_n$ that has the following properties:

(P1) For every $j \in [n]$, where $|V_j| = \infty$, for every vertex $u \in U$, there are infinitely many vertices in V_j that are *not* adjacent to u via any E_i-edge.

(P2) Similarly, for every $j \in [m]$ where $|U_j| = \infty$, for every vertex $v \in V$, there are infinitely many vertices in U_j that are not adjacent to v via any E_i-edge.

An infinite set V_j/U_j that satisfies (P1)/(P2) is called a *strongly infinite* set in G. An infinite $A|B$-biregular graph G is called *strongly partitioned,* if it has a witness partition whose infinite sets are all strongly infinite.

Note that G_0 is an infinite $A_{R,*}|B_{R,*}$-biregular graph and every vertex has a finite degree. Thus G_0 is already strongly partitioned.

There are two cases to consider, depending on whether both $F_{\text{nz-left}}$ and $F_{\text{nz-right}}$ are not empty, or at least one of $F_{\text{nz-left}}$ and $F_{\text{nz-right}}$ is empty. We first consider the case when both $F_{\text{nz-left}}$ and $F_{\text{nz-right}}$ are not empty.

Let $X_0 \uplus X_1$ and $Y_0 \uplus Y_1$ be the partition of $U^{F_{\text{nz-left}},\infty}$ and $V^{F_{\text{nz-right}},\infty}$ in Claim C.7. There are three steps.

Step 1: Construct the E_i-edges for each color $i \in F_{\text{nz-left}}$, similarly to Lemma B.6.

Step 2: Construct the E_i-edges for each color $i \in F_{\text{nz-right}}$ in a manner symmetric to Step 1.

Step 3: Construct the E_i-edges for each color $i \notin R \cup F_{\text{nz-left}} \cup F_{\text{nz-right}}$.

We detail each of these steps in the next paragraphs.

Step 1: Make the E_i-degree of every vertex correct for every $i \in F_{\text{nz-left}}$. This step is divided into three substeps. The first two are similar to case (a) in Lemma B.6 and the third is needed to leave enough room for the construction of the edges of colors in $F_{\text{nz-right}}$.

(a) Make the E_i-degree of every vertex in $U^{i,\text{nz}}$ correct for every $i \in F_{\text{nz-left}}$.
For every $u \in U^{i,\text{nz}}$, we ensure that its degree is correct by connecting it via E_i-edges with some nonadjacent vertices from the set $V^{i,\text{nz}} - V^{F_{\text{nz-right}},\infty}$—that is, vertices that are not yet adjacent to u via any E_i-edges where $i \in F_{\text{nz-left}} \cup R$. Note that $U^{i,\text{nz}}$ is finite. Since $V^{i,\text{nz}} - V^{F_{\text{nz-right}},\infty}$ is infinite and G_0 is strongly partitioned, there is an infinite supply of vertices. So such nonadjacent vertices always exist for every vertex $u \in U^{i,\text{nz}}$. We also make sure that when we add the new edges, there are still infinitely many vertices in each V_j that are not yet adjacent to u for every $j \in [m]$, where V_j is infinite (which is possible since V_j is infinite). Thus, the graph stays strongly partitioned.
After this step, the degree of every vertex in V increases by at most 1. That is, $\sum_{i \in F_{\text{nz-left}}} \deg_{E_i}(v)$ is either 0 or 1 for every $v \in V - V^{F_{\text{nz-right}},\infty}$. Note also that the degrees of vertices in $V^{F_{\text{nz-right}},\infty}$ do not increase.

(b) Make the E_i-degree of every vertex in $V^{i,\text{nz}} - V^{F_{\text{nz-right}},\infty}$ correct for every $i \in F_{\text{nz-left}}$.
Since $U^{i,\text{nz}}$ is finite, the E_i-degree of every vertex in $V^{i,\text{nz}}$ is supposed to be finite. Due to the size being extra-big-enough, $U^{i,\infty}$ contains at least $\delta(A, B)$ vertices. So, for every vertex $v \in V^{i,\text{nz}} - V^{F_{\text{nz-right}},\infty}$, we can add "new" E_i-edges to make its E_i-degree correct by connecting it via E_i-edges with vertices in $U^{i,\infty}$ for every $i \in F_{\text{nz-left}}$. Note that by definition, for every $i \in F_{\text{nz-left}}$, vertices in $U^{i,\infty}$ have ∞ E_i-degrees. So the new E_i-edges in this step will violate their degree requirement.

(c) Make the E_i-degree of every vertex in $V^{F_{\text{nz-right}},\infty}$ correct for every $i \in F_{\text{nz-left}}$.
Here it is useful to recall that for every $i \in F_{\text{nz-left}}$, every vertex in $V^{F_{\text{nz-right}},\infty}$ is supposed to have finite E_i-degree since $U^{i,\text{nz}}$ is finite. This step is similar to step (b), except that we connect via E_i-edges the vertices in Y_0 to some vertices in X_0, and the vertices in Y_1 to some vertices in X_1 for every $i \in F_{\text{nz-left}}$. Since $X_0 \cap U^{i,\infty}$ and $X_1 \cap U^{i,\infty}$ contains at least $\delta(A, B)$ vertices,

there are enough vertices in $X_0 \cap U^{i,\infty}$ and $X_1 \cap U^{i,\infty}$ that we may connect each $v \in V^{F_{\text{nz-right}},\infty}$ with to make the E_i-degree of v correct for every $i \in F_{\text{nz-left}}$. After this step, vertices in X_0 are not adjacent via E_i-edges to vertices in Y_1 for every $i \in F_{\text{nz-left}}$. Similarly, vertices in X_1 are not adjacent via E_i-edges to vertices in Y_0 for every $i \in F_{\text{nz-left}}$. This observation will be important in the next step. See Figure 10 for an illustration.

Step 2: Make the E_i-degree of every vertex correct for every $i \in F_{\text{nz-right}}$. This step consists of three Steps (2(a))–(2(c)) which are symmetric to Steps (1(a))–(1(c)), where the role of $U^{i,\text{nz}}$ is replaced by $V^{i,\text{nz}}$, $U^{i,\infty}$ by $V^{i,\infty}$, and $V^{F_{\text{nz-right}},\infty}$ by $U^{F_{\text{nz-left}},\infty}$. The difference is in Step (1(c)). To make the E_i-degree of vertices in $U^{F_{\text{nz-left}},\infty}$ correct, for every $i \in F_{\text{nz-right}}$, we connect the vertices in X_0 to some vertices in Y_1, and the vertices in X_1 to some vertices in Y_0. Here it is important that vertices in X_0 are not adjacent to vertices in Y_1 via E_i-edges for any $i \in F_{\text{nz-left}}$. Since $Y_1 \cap V^{i,\infty}$ contains at least $\delta(A,B)$ vertices, there are still enough vertices in Y_1 that can be connected to each $u \in X_0$ to make the E_i-degree of u correct for every $i \in F_{\text{nz-right}}$.

Step 3: Make the E_i-degree of every vertex correct for every $i \notin R \cup F_{\text{nz-left}} \cup F_{\text{nz-right}}$. This step is similar to case (c) in Lemma B.6. Let $u_1, u_2, \dots$ and $v_1, v_2, \dots$ be an enumeration of the vertices in U and V. After Step 2, the graph G is still strongly partitioned. In the following we fix a color $i \notin R \cup F_{\text{nz-left}} \cup F_{\text{nz-right}}$. We make the E_i-degree of each vertex u_ℓ and v_ℓ correct, where ℓ ranges from 1 to ∞. We work by induction on ℓ, where the inductive invariant is that after the ℓth iteration, the E_i-degrees of $u_l, v_1, \dots, u_\ell, v_\ell$ are already correct. The process is as follows:

- We pick some vertices in $V^{i,\text{nz}}$ that are not yet adjacent to u_ℓ via any E-edges. We call these vertices the nonadjacent vertices and we pick some of them and connect them to u_ℓ via E_i-edges to make the E_i-degree of u_ℓ correct. For

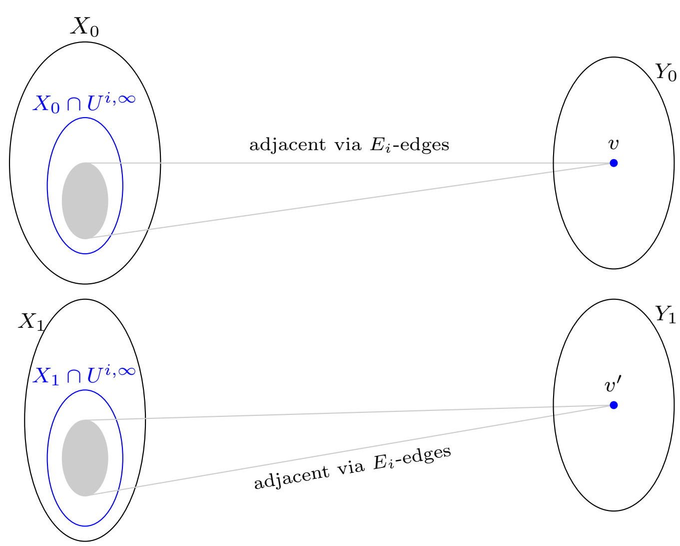

FIG. 10. *An illustration for the construction of edges between $X_0 \cup X_1$ and $Y_0 \cup Y_1$. For each color $i \in F_{nz\text{-}left}$, each $v \in Y_0$ is connected by E_i-edges only to vertices in $X_0 \cap U^{i,\infty}$ and each $v' \in Y_1$ is connected by E_i-edge only to vertices in $X_1 \cap U^{i,\infty}$. This makes vertices in X_0 not connected via E_i-edges to any vertex in Y_1 for any color $i \in F_{nz\text{-}left}$. Likewise, vertices in X_1 are not connected via E_i to any vertex in Y_0 for any color $i \in F_{nz\text{-}left}$. This leaves some "space" for the construction of E_i-edges for color $i \in F_{nz\text{-}right}$, where the vertices in X_0 will be connected to some vertices in Y_1 and the vertices in X_1 to some vertices in Y_0. Note: color appears only in the online article.*

this purpose, we can choose any vertices that are not $v_1, \ldots, v_\ell$ and are not adjacent to any of $u_1, \ldots, u_\ell$. Such vertices always exist, since the graph G is strongly partitioned.

Note that if the E_i-degree of u_ℓ is supposed to be infinite, we have to pick infinitely many nonadjacent vertices. So when we pick these vertices, we also make sure that there are still infinitely many vertices in each V_j that are still not adjacent to all $u_1, \ldots, u_\ell$, for every $j \in [m]$, where V_j is infinite. Thus, the graph is strongly partitioned.

- Similarly, we pick nonadjacent vertices in $U^{i,\mathrm{nz}}$ and connect them to v_ℓ to make the E_i-degree of v_ℓ correct, where nonadjacent vertices are those in $u_1, \ldots, u_\ell$ are not adjacent to any of $v_1, \ldots, v_\ell$.

  Again, such nonadjacent vertices always exist since the graph is strongly partitioned, and we can always pick the new vertices so that the graph stays strongly partitioned after this iteration.

We perform the iteration above for each color $i \notin R \cup F_{\text{nz-left}} \cup F_{\text{nz-right}}$. This completes the construction of an $A|B$-biregular graph G with size $\bar{M}|\bar{N}$ and witness partition $U = U_1 \uplus \cdots \uplus U_m$ and $V = V_1 \uplus \cdots \uplus V_n$ for the case when both $F_{\text{nz-left}}$ and $F_{\text{nz-right}}$ are not empty. For the case when $F_{\text{nz-left}} = \emptyset$, we can do as above, but skip Step 1. Reasoning along the same lines, for the case when $F_{\text{nz-right}} = \emptyset$, we can skip Step 2. This completes the proof of Lemma C.6. ∎

Remark C.8. Note that in the construction of the $A|B$-biregular graph G in Lemma C.6, we construct the E_i-edges for every $i \notin R$ by iterating on every vertex in G. On each iteration, we preserve the "strongly partitioned" property of the graph G. This implies that for every finite subset W of vertices in G, we have

- for every V_j such that V_j is infinite, there are infinitely many vertices in V_j that are not adjacent to any vertex in W;
- similarly, for every U_j such that U_j is infinite, there are infinitely many vertices in U_j that are not adjacent to any vertex in W.

This property will be useful in section D when the completeness requirement is enforced.

To wrap up section C.2, we define formula Gen-$\Psi_{A|B}(\bar{x}, \bar{y})$ for simple matrices A and B as follows,

$$\bigvee_{R \subseteq [t]} \text{Gen-}\Psi^{4,R}_{A|B}(\bar{x}, \bar{y}), \tag{C.8}$$

where each Gen-$\Psi^{4,R}_{A|B}(\bar{x}, \bar{y})$ is defined in (C.5). This formula Gen-$\Psi_{A|B}(\bar{x}, \bar{y})$ captures all the big-enough sizes of $A|B$-biregular graphs, as stated formally as Lemma C.9.

LEMMA C.9. *For every pair of simple matrices A, B, and for each pair of size vectors $\bar{M}, \bar{N}$ with $\bar{M}|\bar{N}$ extra-big-enough for $A|B$, the following holds: the formula Gen-$\Psi_{A|B}(\bar{M}, \bar{N})$ holds in $\mathcal{N}_\infty$ if and only if there is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$.*

Proof. Let A and B be simple matrices with t rows and $\bar{M}|\bar{N}$ be big-enough for $A|B$. Suppose there is an $A|B$-biregular graph $G = (U, V, E_1, \ldots, E_t)$ with size $\bar{M}|\bar{N}$. Let $R = \{i :$ every vertex in V has finite E_i-degree$\}$. By Lemma C.6, $\Psi^{4,R}_{A|B}(\bar{M}, \bar{N})$ holds. Thus, Gen-$\Psi^{4}_{A|B}(\bar{M}, \bar{N})$ holds in $\mathcal{N}_\infty$.

Conversely, suppose Gen-$\Psi_{A|B}(\bar{M}, \bar{N})$ holds in $\mathcal{N}_\infty$. Let R be such that

Gen-$\Psi^{4,R}(\bar{M},\bar{N})$ holds. By Lemma C.6, there is an $A|B$-biregular graph with size $\bar{M}|\bar{N}$, where $R=\{i:\text{every vertex in } V \text{ has finite } E_i\text{-degree}\}$. ◻

The remark below will be useful for the complexity analysis later on.

Remark C.10. Every formula Gen-$\Psi^{4,R}_{A|B}(\bar{x},\bar{y})$ is a disjunction of the formulas Gen-$\Psi^{3,R'}_{A_{R,*}|B_{R,*}}(\bar{x},\bar{y})$ for every subset R' of the rows in $A_{R,*}|B_{R,*}$. In turn, each formula Gen-$\Psi^{3,R'}_{A_{R,*}|B_{R,*}}(\bar{x},\bar{y})$ is a disjunction of the formulas Gen-$\Psi^{2,R''}_{A_{R',*}|B_{R',*}}(\bar{x},\bar{y})$ for every subset R'' of the rows in $A_{R',*}|B_{R',*}$. Pulling out all the disjunctions, the formula Gen-$\Psi_{A|B}(\bar{x},\bar{y})$ can be written as a disjunction $\bigvee_i \psi_i(\bar{x},\bar{y})$, where each $\psi_i(\bar{x},\bar{y})$ is a conjunction of $O(t)$ equations and inequations (for short, "conjunction of (in)equations").

C.3. Encoding of not extra-big-enough components for simple matrices. The encoding of not extra-big-enough for the general case is a routine adaptation of the one in subsection 5.4. We omit the details. Disjoining this to the extra-big-enough formula completes the description for simple matrices without the completeness requirement.

The remark below will also be used in the complexity analysis later on.

Remark C.11. Let t be the number of rows in matrices A and B. By Remark C.10, the formula Gen-$\Psi_{A|B}(\bar{x},\bar{y})$ is a disjunction of conjunctions of $O(t)$ (in)equations. By an adaptation of Remark 5.12, each formula $\Phi_i(\bar{x},\bar{y})$ for the not extra-big-enough cases is a disjunction of conjunctions of $O(t^4\delta(A,B)^4)$ (in)equations. Thus, the formula $\mathsf{bireg}_{A|B}(\bar{x},\bar{y})$, which is a disjunction of Gen-$\Psi_{A|B}(\bar{x},\bar{y})$ and all the $\Phi_i(\bar{x},\bar{y})$'s, can be written as a disjunction of conjunctions of $O(t^4\delta(A,B)^4)$ (in)equations.

Appendix D. The extension of section 6 to the general case. In section 6 we constructed a formula that captures all possible finite sizes of *complete* $A|B$-biregular graphs, where A and B are simple degree matrices. There we argued that it was sufficient to consider only the case when $A|B$ is a good pair (as defined in Definition 6.1), since otherwise there are only some fixed number of possible sizes of $A|B$-biregular graphs, which can be enumerated.

In this appendix we will extend the formulas in section 6 to the case where graphs may be infinite. As in section 6, we only need to consider the cases where $A|B$ is a good pair. There are two new cases to consider. The first (subsection D.1) concerns graphs where both sides have infinitely many vertices, while in the second (subsection D.2) there are infinitely many vertices on exactly one side.

D.1. The case when both sides have infinitely many vertices. Let A be a matrix with t rows. We write $\mathsf{col}_\infty(A_{*,j})$ to denote the set $\{i: A_{i,j}=\infty\}$. For $R\subseteq[t]$, we let $J(R,A)=\{j:\mathsf{col}_\infty(A_{*,j})=R\}$.

DEFINITION D.1. *Let A and B be simple matrices with t rows. Let m and n be the number of columns of A and B. For size vectors $\bar{M}=(M_1,\dots,M_m)$ and $\bar{N}=(N_1,\dots,N_n)$, we say that $\bar{M}|\bar{N}$* is a good color size for $A|B$ *if for every $R\subseteq[t]$, we have*

- $\|\bar{M}^T\|_{J(R,A)}=0$ *or* $\geqslant\delta(A,B)+1$,
- $\|\bar{N}^T\|_{J(R,B)}=0$ *or* $\geqslant\delta(A,B)+1$.

We will show that a "good color size" of a complete $A|B$-biregular graph implies a certain property of the matrices A and B which will be useful later on.

LEMMA D.2. *Let $A|B$ be a pair of simple matrices with t rows. Suppose there is a complete $A|B$-biregular graph $G=(U,V,E_1,\dots,E_t)$ with size $\bar{M}|\bar{N}$, where $\|\bar{M}^T\|=\|\bar{N}^T\|=\infty$ and $\bar{M}|\bar{N}$ is a good color size for $A|B$. Then*

- *for every vertex $u\in U$, there is $i\in[t]$ such that $\deg_{E_i}(u)=\infty$ and row i in B contains a periodic entry;*
- *for every vertex $v\in V$, there is $i'\in[t]$ such that $\deg_{E_{i'}}(v)=\infty$ and row i' in A contains a periodic entry.*

Note that since A and B are simple matrices and ∞ is regarded as a periodic entry, the conclusion implies that row i and i' in A and B can contain *only* periodic entries. So the lemma implies that $A|B$ is a good pair of simple matrices.

Proof. We first prove an easy combinatorial claim.

CLAIM D.3. *Let $\mathcal{U}$ be an infinite set and let $\mathcal{Z}$ be a (not necessarily finite) family of subsets of $\mathcal{U}$ such that every $Z\in\mathcal{Z}$ is cofinite in $\mathcal{U}$ (i.e., $\mathcal{U}-Z$ is finite). Then every finite subset of $\mathcal{Z}$ has a nonempty intersection.*

Proof. Let $Z_1,\dots,Z_n\in\mathcal{Z}$. By de Morgan's law, $\bigcap_{j=1}^n Z_j=\mathcal{U}-(\bigcup_{j=1}^n \mathcal{U}-Z_j)$. Since each Z_j is cofinite in $\mathcal{U}$, the claim follows immediately. ∎

We prove the first bullet item of the lemma, with the second proven analogously. Let $U=U_1\uplus\cdots\uplus U_m$ and $V=V_1\uplus\cdots\uplus V_n$ be the witness partition. For a vertex $u\in U$ in G, let $\Gamma(u)$ denote the set of vertices adjacent to u by some E_i-edge, where the E_i-degree of u is infinite. For each $u\in U$, every element of V is connected to u by some E_i-edge (since G is a complete bipartite graph), and thus the number of elements of V not in $\Gamma(u)$ is finite.

Suppose $u\in U_j$ for some $j\in[m]$ and let $R=\mathsf{col}_\infty(A_{*,j})$. Since $\bar{M}|\bar{N}$ is a good color size for $A|B$, we have $\|\bar{M}^T\|_{J(R,A)}\geqslant\delta(A,B)+1$. We pick k vertices $u_1,\dots,u_k\in\bigcup_{j\in J(R,A)}U_j$, where $u_1=u$ and $k\geqslant\delta(A,B)+1$. By the claim above, there is a vertex v in the intersection $\bigcap_{j\in[k]}\Gamma(u_j)$. This means v is adjacent to all vertices $u_1,\dots,u_k$ via some E_i-edges, where $i\in R$. Since $k>\|\text{fin-offset}(B)\|$, there is $E_i\in R$, where $\deg_{E_i}(v)$ is a periodic entry of B. Since B is a simple matrix, this implies that row i in B contains only periodic entries. Note that $E_i\in R$, so the E_i-degree of u is ∞. This completes the proof of the first bullet item. ∎

Let A and B be simple matrices with m and n columns, respectively. We denote by $(\mathcal{C}1)$ and $(\mathcal{C}2)$ the following constraints:

($\mathcal{C}1$) For every $j\in[m]$ where $M_j\neq 0$, there is a color $i\in[t]$ such that $A_{i,j}=\infty$ and $B_{i,*}$ contains only periodic entries.

($\mathcal{C}2$) For every $j\in[n]$ where $N_j\neq 0$, there is a color $i\in[t]$ such that $B_{i,j}=\infty$ and $A_{i,*}$ contains only periodic entries.

Note that both $(\mathcal{C}1)$ and $(\mathcal{C}2)$ are Presburger definable and formalize the properties from items (1) and (2) in Lemma D.2.

We define $\xi^{(1)}_{A|B}(\bar{x},\bar{y})$, where $\bar{x}=(x_1,\dots,x_m)$ and $\bar{y}=(y_1,\dots,y_n)$, as follows:

$$\mathsf{bireg}_{A|B}(\bar{x},\bar{y})\wedge\|\bar{x}^T\|=\|\bar{y}^T\|=\infty\ \wedge\ (\mathcal{C}1)\ \wedge\ (\mathcal{C}2) \tag{D.1}$$

$$\wedge\bigwedge_{R\subseteq[t]}\Big(\|\bar{x}^T\|_{J(R,A)}=0\ \vee\ \|\bar{x}^T\|_{J(R,A)}\geqslant\delta(A,B)+1\Big) \tag{D.2}$$

$$\wedge\bigwedge_{R\subseteq[t]}\Big(\|\bar{y}^T\|_{J(R,B)}=0\ \vee\ \|\bar{y}^T\|_{J(R,B)}\geqslant\delta(A,B)+1\Big). \tag{D.3}$$

Above $\mathsf{bireg}_{A|B}(\bar{x},\bar{y})$ is the characterizing formula for not-necessarily-complete biregular graphs. Intuitively, (D.2) and (D.3) state that $\bar{x}|\bar{y}$ is a good color size.

LEMMA D.4. *For every pair of simple matrices A, B and for every pair of size vectors $\bar{M}$, $\bar{N}$, the formula $\xi^{(1)}_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}_\infty$ if and only if there is a complete $A|B$-biregular graph of size $\bar{M}|\bar{N}$, where $\|\bar{M}^T\| = \|\bar{N}^T\| = \infty$ and $\bar{M}|\bar{N}$ is a good color size for $A|B$.*

Proof. That $\xi^{(1)}_{A|B}(\bar{M},\bar{N})$ holding is necessary follows from Lemma D.2. Now we show that it is also a sufficient condition. Suppose $\xi^{(1)}_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}_\infty$, which implies there is a (not-necessarily-complete) $A|B$-biregular graph $G = (U, V, E_1, \ldots, E_t)$ with size $\bar{M}|\bar{N}$. Let $U = U_1 \uplus \cdots \uplus U_m$ and $V = V_1 \uplus \cdots \uplus V_n$ be the witness partition. By Remark C.8, the graph G has the following property:

($\mathcal{P}$) *For every finite subset $W \subseteq U$, there are infinitely many vertices in V that are not adjacent to any vertex in W.*
While for every finite subset $W \subseteq V$, there are infinitely many vertices in U that are not adjacent to any vertex in W.

We enumerate the elements $u_1, u_2, \ldots$ and $v_1, v_2, \ldots$ in U and V, respectively. We will make G a complete $A|B$-biregular graph by iterating through all $\ell = 1, 2, \ldots$, where on each iteration ℓ, we first add "new" edges so that u_ℓ is adjacent to all the vertices $v_\ell, v_{\ell+1}, \ldots$ and then some more "new" edges so that v_ℓ is adjacent to all the vertices $u_{\ell+1}, u_{\ell+2}, \ldots$. These new edges will preserve the $A|B$-biregularity of the graph G and as the iteration index ℓ goes to ∞, the graph G becomes complete.

Before we proceed to the construction, we first explain the main idea behind making u_ℓ adjacent to all the vertices $v_\ell, v_{\ell+1}, \ldots$. Choose $i_0 \in [t]$ such that the E_{i_0}-degree of u_ℓ is ∞ and the row $B_{i_0,*}$ contains only periodic entries—such i_0 exists due to ($\mathcal{C}1$). We add new E_{i_0}-edges so that

(a) u_ℓ is adjacent to all the vertices $v_\ell, v_{\ell+1}, \ldots$ (that are not yet adjacent to u_ℓ) via E_{i_0}-edges;
(b) for every vertex $v_h \in \{v_\ell, v_{\ell+1}, \ldots\}$
  - if the E_{i_0}-degree of v_h is not ∞, the new E_{i_0}-edges increase it by p;
  - if the E_{i_0}-degree of v_h is ∞, there are either 1 or p new E_{i_0}-edges adjacent to v_h; in particular the degree is still infinite;

(c) for every vertex $u_h \in \{u_{\ell+1}, v_{\ell+2}, \ldots\}$, either there are no new E_{i_0}-edges added, or the E_{i_0}-degree increases by a multiple of p.

Adding new edges to make v_ℓ adjacent to all the vertices $u_{\ell+1}, u_{\ell+2}, \ldots$ can be done in the same manner. The purpose of (a) is to make G complete while the purpose of (b) and (c) is to preserve the $A|B$-biregularity of G.

Since U (resp., V) is countable, every vertex $u \in U$ (resp., $v \in V$) has a finite index ℓ such that $u_\ell = u$ (resp., $v_\ell = v$). After the ℓth iteration we do not add any more edges adjacent to u_ℓ and v_ℓ. Therefore, for every vertex $w \in U \cup V$ for every color $i \in [t]$, if $\deg_{E_i}(w)$ is finite in the original graph G, it stays finite as the iteration index ℓ goes to ∞. If $\deg_{E_i}(u)$ is ∞ in the original graph G, it stays ∞, since we are only adding edges. Thus, if the original graph G is $A|B$-biregular, as the iteration index ℓ goes to ∞, the resulting graph is still $A|B$-biregular.

Note also that due to (b) and (c), after the ℓth iteration, the degree of every vertex in $\{u_{\ell+1}, u_{\ell+2}, \ldots\} \cup \{v_{\ell+1}, v_{\ell+2}, \ldots\}$ increases only by some finite number, i.e., by 0, 1, or a multiple of p. Thus, property ($\mathcal{P}$) still holds for every finite subset $W \subseteq \{u_{\ell+1}, u_{\ell+2}, \ldots\} \cup \{v_{\ell+1}, v_{\ell+2}, \ldots\}$ in the sense that

> for every finite subset $W \subseteq \{u_{\ell+1}, u_{\ell+2}, \ldots\}$, there are infinitely many vertices in V that are not adjacent to any vertex in W. While for every finite subset $W \subseteq \{v_{\ell+1}, v_{\ell+2}, \ldots\}$, there are infinitely many vertices in U that are not adjacent to any vertex in W.

We will call this *the nonadjacency invariant.*

We devote the rest of the proof to the details on how to add edges adjacent to vertex u_ℓ. The argument for vertex v_ℓ is handled symmetrically. Let u_ℓ be a vertex in U_j. By constraint $(\mathcal{C})$, there is some color $i_0 \in [t]$, where $A_{i_0,j} = \infty$ and $B_{i_0,*}$ contains only periodic entries. Since A is a simple matrix, row i_0 in A also contains periodic (possibly infinite) entries. We will add E_{i_0}-edges so that u_ℓ is adjacent to every vertex in V. Note, however, that some care is needed, since the E_{i_0}-degree of some vertices—those with finite E_{i_0}-degree bound—can only increase by a multiple of p.

Let Z denote the set of vertices in V that are not adjacent to vertex u_ℓ. By the nonadjacency invariant, the set Z is infinite. Let $Z = Z_{\text{fin}} \cup Z_\infty$ be a partition of Z where every vertex in Z_{fin} has finite E_{i_0}-degree and every vertex in Z_∞ has infinite E_{i_0}-degree.

First, we add E_{i_0}-edges between u_ℓ and every vertex in Z. At this point, vertex u_ℓ is already adjacent to every vertex in V. Note that the E_{i_0}-degree of each vertex in Z_∞ stays infinite. However, the E_{i_0}-degree of vertices in Z_{fin} increases by 1. So we need to add additional edges to make it increase further by $(p-1)$. There are two cases.

- *Case* 1: $|Z_{\text{fin}}|$ is finite. Since the set Z is infinite, we infer that $|Z_\infty|$ is infinite. Let Y be a finite subset of Z_∞ so that the sum $|Z_{\text{fin}}| + |Y|$ is some multiple of p. By the nonadjacency invariant, there are infinitely many vertices in U that are not adjacent to any vertex in $Z_{\text{fin}} \cup Y$. We pick $(p-1)$ such vertices $w_1, \ldots, w_{p-1}$ and add E_{i_0}-edges for every pair in $\{w_1, \ldots, w_{p-1}\} \times (Z_{\text{fin}} \cup Y)$. That is, $\{w_1, \ldots, w_{p-1}\} \times (Z_{\text{fin}} \cup Y)$ becomes a complete bipartite graph of E_{i_0}-edges. Note that the E_{i_0}-degrees of vertices $w_1, \ldots, w_{p-1}$ increase by a multiple of p, since $|Z_{\text{fin}} \cup Y|$ is a multiple of p. Moreover, the E_{i_0}-degrees of vertices in Z_{fin} increase further by $(p-1)$. The E_{i_0}-degrees of vertices in Y remain infinite. Thus, after this construction G is still $A|B$-biregular. See Figure 11 for an illustration.

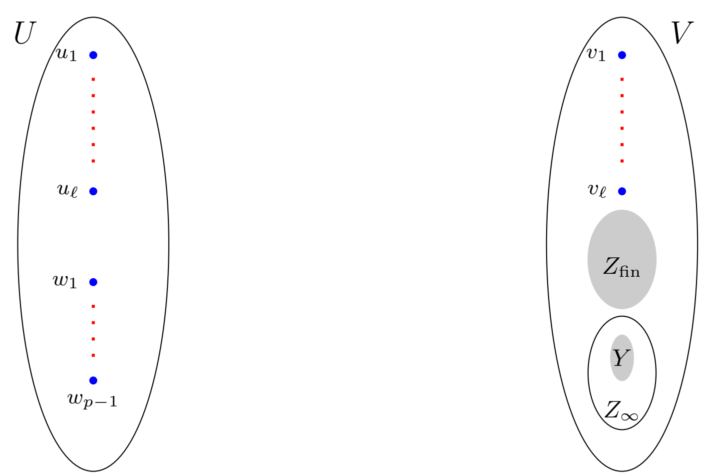

FIG. 11. *An illustration for the choices of $w_1, \ldots, w_{p-1}$ and $Y \subseteq Z_\infty$ to construct the complete $A|B$-biregular graph when $|Z_{fin}|$ is finite. First, we connect u_ℓ with all the vertices in $Z_{fin} \cup Z_\infty$ via an E_{i_0}-edge. Then, to ensure the degrees of vertices in Z_{fin} increase by a multiple of p, we pick $w_1, \ldots, w_{p-1}$ and connect them via E_{i_0}-edges with all the vertices in $Z_{fin} \cup Y$, where $Y \subseteq Z_\infty$ such that $|Z_{fin}| + |Y|$ is a multiple of p. Note: color appears only in the online article.*

- *Case* 2: $|Z_{\text{fin}}|$ is infinite. We partition Z_{fin} into infinitely many pairwise disjoint sets $Z_1 \uplus Z_2 \uplus \cdots$, where $|Z_h| = p$ for each $h = 1, 2, \ldots$. We increase the E_{i_0}-degrees of vertices in each Z_h by iterating the following process for each $h = 1, 2, \ldots$: we pick a finite set $X \subseteq U$ such that $|X| = p - 1$ and every vertex in X is not adjacent to any vertex in $Z_1 \cup \cdots \cup Z_h$. Such a set X exists, by the nonadjacency invariant. Then we add E_{i_0}-edges between every pair in $Z_h \times X$. That is, $Z_h \times X$ becomes a complete bipartite graph of E_{i_0}-edges. See Figure 12 for an illustration. After this construction, the E_{i_0}-degree of each vertex in each Z_h increases further by $(p - 1)$. Since each $|Z_h| = p$, we also increase the E_{i_0}-degrees of some vertices in U by p. Thus, G is still $A|B$-biregular. ∎

Lemmas D.4 and D.13 deal with all the $\bar{M}|\bar{N}$ that are good color sizes for $A|B$. To capture the sizes that are not good color sizes, we can use fixed size encoding, as in subsection 5.4. Note that if $\bar{M}|\bar{N}$ is not a good color size for $A|B$, there is $R \subseteq [t]$ such that

$$1 \leqslant \|\bar{x}^T\|_{J(R,A)} \leqslant \delta(A,B) \qquad \text{or} \qquad 1 \leqslant \|\bar{y}^T\|_{J(R,B)} \leqslant \delta(A,B).$$

Thus, we can fix $\|\bar{x}^T\|_{J(R,A)}$ or $\|\bar{y}^T\|_{J(R,B)}$ to some r, where $1 \leqslant r \leqslant \delta(A,B)$. Recall that $J(R,A)$ is a subset of columns of A, while $J(R,B)$ is a subset of the columns of B. Thus in fixing one of these norms, we are focusing on complete $A|B$-biregular graphs $G = (U, V, E_1, \ldots, E_t)$ with sizes $\bar{M}|\bar{N}$, where the sum of some components in $\bar{M}$ (or $\bar{N}$) is fixed to $r \leqslant \delta(A,B)$. For example, we can define the formula $\Phi^r_{A|B}(\bar{x}, \bar{y})$ such that for every $\bar{M}|\bar{N}$, $\Phi^r_{A|B}(\bar{M}, \bar{N})$ holds if and only if there is a complete $A|B$-biregular graph with size $\bar{M}|\bar{N}$, where $\|\bar{x}^T\|_{J(R,A)} = r$. The construction of $\Phi^r_{A|B}(\bar{x}, \bar{y})$ is very similar to the one in section 5.4, so we omit it.

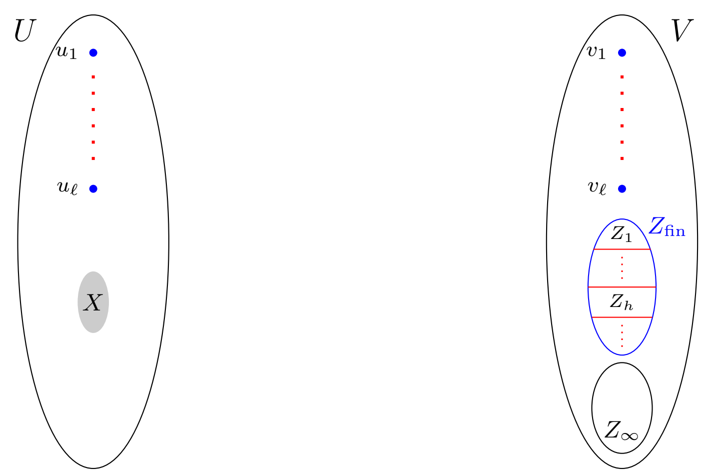

FIG. 12. *An illustration for the choices of $w_1, \ldots, w_{p-1}$ and $Y \subseteq Z_\infty$ to construct the complete $A|B$-biregular graph when $|Z_{fin}|$ is infinite. First, we connect u_l with all the vertices in $Z_{fin} \cup Z_\infty$ via an E_{i_0}-edge, thus, increasing the E_i-degree of vertices in $Z_{fin} \cup Z_\infty$ by 1. Then, we partition Z_{fin} into $Z_1 \uplus Z_2 \uplus \cdots$, where each Z_h has cardinality p. To make sure that the E_{i_0}-degrees in Z_{fin} increase by a multiple of p, for each $h = 1, 2, \ldots$, we pick a set $X \subseteq U$ s.t. $|X| = p - 1$ and every vertex in X is not adjacent to any vertex in $Z_1 \cup \cdots \cup Z_h$. Then, we connect every vertex in X with every vertex in Z_h via E_{i_0}-edges. Note: color appears only in the online article.*

To wrap up this subsection, we define the formula $\mathsf{c\text{-}bireg}^{\infty,\infty}_{A|B}(\bar{x},\bar{y})$ for simple matrices A and B as follows,

$$\xi^{(1)}_{A|B}(\bar{x},\bar{y}) \ \vee\ \varphi(\bar{x},\bar{y}) \ \vee\ \bigvee_j \phi_j(\bar{x},\bar{y}), \tag{D.4}$$

where $\xi^{(1)}_{A|B}(\bar{x},\bar{y})$ is defined in (D.1)–(D.3), $\varphi(\bar{x},\bar{y})$ captures all the sizes $\bar{M}|\bar{N}$ that are not good color sizes for $A|B$, and the disjunction $\bigvee_j \phi_j(\bar{x},\bar{y})$ enumerates all possible sizes $\bar{M}|\bar{N}$ when $A|B$ is not a good pair. By Remark 6.2, when $A|B$ is not a good pair, complete $A|B$-biregular graphs can only have sizes $\bar{M}|\bar{N}$, where $\|\bar{M}^T\|+\|\bar{N}^T\| \leqslant 2\delta(A,B)$. It is clear that this remark holds regardless of whether an ∞ entry is allowed. Since there are only finitely many sizes satisfying this upper bound, they can be enumerated. The formula $\mathsf{c\text{-}bireg}^{\infty,\infty}_{A|B}(\bar{x},\bar{y})$ captures the sizes of all possible $A|B$-biregular graphs where both sides have infinitely many vertices.

Remark D.5. We will again make some further observations that will be important only for the complexity analysis. Suppose t is the number of rows in matrices A and B. By Remark C.11, $\xi^{(1)}_{A|B}(\bar{x},\bar{y})$ is a disjunction of conjunctions of $O(t^4\delta(A,B)^4)$ (in)equations.

As in Remark 5.12, the encoding of components of a fixed size r yields $O(rt)$ (in)equations. Since $r \leqslant \delta(A,B)$ and there are 2^t subsets $R \subseteq [t]$, the formula for the *fixed size encoding* can be written as a disjunction of conjunctions of $O(2^t t\delta(A,B))$ (in)equations. So, the whole formula $\mathsf{c\text{-}bireg}^{\infty,\infty}_{A|B}(\bar{x},\bar{y})$ can be rewritten as a disjunction of conjunctions of $O(2^t t^4 \delta(A,B)^4)$ (in)equations.

D.2. The case when exactly one side has only finitely many vertices. In this subsection we will give the formula that captures the sizes of all possible $A|B$-biregular graphs, where on the left-hand side there are infinitely many vertices and on the right-hand side there are only finitely many vertices. Here the degree matrices A and B can be arbitrary degree matrices, i.e., we drop the assumption that they must be simple matrices.

In a first step (subsection D.2.1) we consider the case where the degree matrix B is restricted to a very special form and the size vectors on the left contain only ∞. In a second step (subsection D.2.2) we show that capturing the sizes of $A|B$-biregular graphs, where exactly one side has infinitely many vertices, can be reduced to the finite case and the case in subsection D.2.1.

D.2.1. A special case. We fix matrices A and B (with t rows) with the following properties:

- A contains only finite entries.
- Each entry in B is either 0 or ∞.
- Every row and every column in B has ∞ entry.

We note that for such A and B, in a complete $A|B$-biregular graph it is necessary that the left side has infinitely many vertices and the right side has only finitely many. We will define a formula that captures all possible size vectors $\bar{N}$, where $\|\bar{N}^T\| \neq \infty$ and there is a complete $A|B$-biregular graph with size $(\infty,\ldots,\infty)|\bar{N}$.

Let m and n be the number of columns in A and B. We start with a simple observation.

Remark D.6. Let $G=(U,V,E_1,\ldots,E_t)$ be a complete $A|B$-biregular graph with witness partition $U=U_1 \uplus \cdots \uplus U_m$ and $V=V_1 \uplus \cdots \uplus V_n$. Let $u \in U$ and let j be the index such that $u \in U_j$.

For each $i \in [t]$, let Z_i be the set of vertices adjacent to u via E_i-edges. Since G is complete, $Z_1 \uplus \cdots \uplus Z_t$ partitions the set V. Moreover, since G is $A|B$-biregular, for every $i \in [t]$,

- $|Z_i| = E_i$-degree of $u = A_{i,j}$;
- every vertex in Z_i has ∞ E_i-degree.
  Recall that every entry in B can only be either 0 or ∞, hence, the E_i-degree of every vertex in V can only be 0 or ∞.

We will call the partition $Z_1 \uplus Z_2 \uplus \cdots \uplus Z_t$, *the partition of* V according to u. As we will see later, we can construct a Presburger formula that defines the sizes of the partitions of V according to vertices in U_j for every vertex in U_j.

For $j \in [m]$ for each $k \in [t]$, define the formula $\varphi_{k,j}(z_1, \ldots, z_k, s_1, \ldots, s_n)$ inductively on k as follows:

- When $k = 1$, $\varphi_{1,j}(z_1, s_1, \ldots, s_n)$ is given by

$$z_1 = s_1 + \cdots + s_n = A_{1,j} \ \wedge \bigwedge_{h \in [n]} s_h \neq 0 \to B_{1,h} = \infty.$$

- When $k \geqslant 2$, $\varphi_{k,j}(z_1, \ldots, z_k, s_1, \ldots, s_n)$ is given by

$$\begin{aligned} \exists c_1 \cdots \exists c_n \qquad & c_1 + \cdots + c_n = z_k \wedge \bigwedge_{h \in [n]} c_h \neq 0 \ \to \ B_{k,h} = \infty \\ & \wedge \varphi_{k-1,j}(z_1, \ldots, z_{k-1}, s_1 - c_1, \ldots, y_n - s_n). \end{aligned}$$

Finally, define the formula $\xi_{A|B}(\bar{y})$, where $\bar{y} = (y_1, \ldots, y_n)$,

$$\xi_{A|B}(\bar{y}) \ := \bigwedge_{j \in [m]} \exists z_1 \cdots \exists z_t \ \varphi_{t,j}(z_1, \ldots, z_t, \bar{y}). \tag{D.5}$$

We will show that $\xi_{A|B}(\bar{y})$ captures all size vectors $\bar{N}$ such that there are complete $A|B$-biregular graph with size $(\infty, \ldots, \infty)|\bar{N}$. The variables $z_1, \ldots, z_t$ in the formula $\varphi_{t,j}(z_1, \ldots, z_t, y_1, \ldots, y_n)$ represent the cardinalities $|Z_1|, \ldots, |Z_t|$ for the partition $Z_1 \uplus \cdots \uplus Z_t$ according to a vertex in U_j. We start with an easy lemma, proven by induction on k.

LEMMA D.7. *For every $k \in [t]$ for every $z_1, \ldots, z_k, s_1, \ldots, s_n \in \mathbb{N}$, if $\varphi_{k,j}(z_1, \ldots, z_k, s_1, \ldots, s_n)$ holds in $\mathcal{N}_\infty$, then $z_1 + \cdots + z_k = s_1 + \cdots + s_n$.*

LEMMA D.8. *For every size vector $\bar{N}$, where $\|\bar{N}^T\| \neq \infty$, the formula $\xi_{A|B}(\bar{N})$ holds in $\mathcal{N}_\infty$ precisely when there is a complete $A|B$-biregular graph with size $(\infty, \ldots, \infty)|\bar{N}$.*

Proof. Let $\bar{N} = (N_1, \ldots, N_n)$ be a size vector where none of $N_1, \ldots, N_n$ are ∞. We first show that $\xi_{A|B}(\bar{N})$ holding in $\mathcal{N}_\infty$ is a necessary condition. Suppose there is a complete $A|B$-biregular graph $G = (U, V, E_1, \ldots, E_t)$ with size $(\infty, \ldots, \infty)|\bar{N}$. Let $U = U_1 \uplus \cdots \uplus U_m$ and $V = V_1 \uplus \cdots \uplus V_n$ be the witness partition.

We will show that for every $j \in [m]$, $\varphi_{t,j}(z_1, \ldots, z_t, \bar{N})$ holds for some $z_1, \ldots, z_t$. To this end, let $j \in [m]$. We pick a vertex $u \in U_j$ and let $Z_1 \uplus \cdots \uplus Z_t$ be the partition of V according to u. Let $z_i = |Z_i|$, for every $i \in [t]$.

The next claim can be proven by straightforward induction on k.

CLAIM D.9. *For every $k \in [t]$, the formula $\varphi_{k,j}(z_1, \ldots, z_k, s_1, \ldots, s_n)$ holds in $\mathcal{N}_\infty$, where $s_h = |V_h \cap (Z_1 \cup \cdots \cup Z_k)|$ for each $h \in [n]$.*

In particular, when $k=t$, $s_h = |V_h \cap (Z_1 \cup \cdots \cup Z_t)| = |V_h| = N_h$ for each $h \in [n]$, since $Z_1 \cup \cdots \cup Z_t = V$. Therefore, $\varphi_{t,j}(z_1,\ldots,z_t,\bar{N})$ holds. Thus, $\xi_{A|B}\bar{N}$ holds.

We now prove that $\xi_{A|B}(\bar{N})$ holding is a sufficient condition. Suppose $\xi_{A|B}(\bar{N})$ holds, where $\bar{N}=(N_1,\ldots,N_n)$. Let $U_1,\ldots,U_m$ be pairwise disjoint infinite sets and let $V_1,\ldots,V_n$ be pairwise disjoint sets, where $|V_h| = N_h$ for each $h \in [n]$.

We will construct a complete $A|B$-biregular graph $G=(U,V,E_1,\ldots,E_t)$ with size $(\infty,\ldots,\infty)|\bar{N}$ and witness partition $U = U_1 \uplus \cdots \uplus U_m$ and $V = V_1 \uplus \cdots \uplus V_n$.

Let $j \in [m]$. Since $\xi_{A|B}(\bar{N})$ holds, there is $z_1,\ldots,z_t$ such that $\varphi_{t,j}(z_1,\ldots,z_t,\bar{N})$ holds.

The following claim is proven by straightforward induction on k.

CLAIM D.10. *For every $k \in [t]$, there are pairwise disjoint sets $Z_1,\ldots,Z_k \subseteq V$ such that for every $i \in [k]$ $z_i = |Z_i| = A_{i,j}$ and $Z_i \subseteq \bigcup_{h\in \inf(B_{i,*})} V_h$.*

In particular, when $k=t$, we have pairwise disjoint sets $Z_1,\ldots,Z_t \subseteq V$ such that for every $i \in [t]$ $z_i = |Z_i| = A_{i,j}$ and $Z_i \subseteq \bigcup_{h\in \inf(B_{i,*})} V_h$. By Lemma D.7, the sum $z_1+\cdots+z_t = |Z_1|+\cdots+|Z_t| = N_1+\cdots+N_n$. Hence, $Z_1 \uplus \cdots \uplus Z_t$ is a partition of V. For every $i \in [t]$, we connect every vertex $u \in U_j$ with every vertex in Z_i via an E_i-edge. Thus after this step, every vertex in U_j is adjacent to every vertex in V.

Note that the E_i-degree of every vertex in U_j is $|Z_i| = A_{i,j}$. Moreover, we connect u with a vertex $v \in V$ only when the E_i-degree of v is supposed to be ∞—since $Z_i \subseteq \bigcup_{h\in \inf(B_{i,*})} V_h$. Thus, the resulting graph is $A|B$-biregular. By repeating the above process for every $j \in [m]$, we obtain a complete $A|B$-biregular graph.

D.2.2. The formula for the case with infinitely many vertices on the left and finitely many vertices on the right. In this subsection we will define the formula that captures precisely the sizes of all possible $A|B$-biregular graphs where the left-hand side has infinitely many vertices and the right-hand side has only finitely many vertices. Here we do not require the degree matrices to be simple matrices—as defined in Definition 5.1.

In the following lemma, we fix degree matrices $A \in \mathbb{N}^{t\times m}_{\infty,+p}$ and $B \in \mathbb{N}^{t\times n}_{\infty,+p}$.

LEMMA D.11. *Suppose $G=(U,V,E_1,\ldots,E_t)$ is a complete $A|B$-biregular graph with witness partition $U_1 \uplus \cdots \uplus U_m$ and $V_1 \uplus \cdots \uplus V_n$. Suppose U is infinite and V is finite. Let $R = \{i \in [t] \mid |E_i| = \infty\}$ and let $J = \{j \in [m] \mid |U_j| = \infty\}$. Then we have the following:*

1. *For every color $i \notin R$ for every $j \in J$, $A_{i,j}$ is 0 or 0^{+p}.*
2. *For every $j \in [n]$, there is $i \in R$ with $B_{i,j} = \infty$.*
3. *For every $i \in R$, the row $B_{i,*}$ contains an ∞ entry.*
4. *For every $j \in J$ for every $i \notin R$, all but finitely many vertices in U_j have zero E_i-degree.*
5. *There are only finitely many vertices in U for which there is a $v \in V$ adjacent to the vertex by an E_i-edge and the E_i-degree of v is finite.*

Proof. To prove (1), let $j \in J$, i.e., the set U_j is infinite. If there were $i \notin R$ such that $A_{i,j} \neq 0$ or $\neq 0^{+p}$, the number of edges in E_i is infinite, which contradicts the assumption that $i \notin R$.

For (2), let $j \in [n]$. Since G is a complete $A|B$-biregular graph, the total degree of each vertex $v \in V_j$ must equal $|U|$, i.e,

$$\sum_{i\in[t]} (E_i\text{-degree of } v) \;=\; \sum_{i\in R} (E_i\text{-degree of } v) \;+\; \sum_{i\notin R} (E_i\text{-degree of } v) \;=\; |U|.$$

By the definition of R, the sum $\sum_{i \notin R}(E_i\text{-degree of } v)$ is finite. Since U is infinite, the sum $\sum_{i \in R}(E_i\text{-degree of } v)$ must be infinite. Therefore, there is $i \in R$ such that $B_{i,j} = \infty$.

For (3), let $i \in R$. The cardinality E_i is $|E_i| = \sum_{v \in V}(E_i\text{-degree of } v)$. Since $i \in R$, the cardinality $|E_i| = \infty$. Thus, there is $v \in V$ with E_i-degree ∞. Therefore, row $B_{i,*}$ must contain ∞.

For (4), let $j \in J$ and $i \notin R$. There can only be finitely many vertices in U_j with nonzero E_i-degree. Otherwise, $|E_i| = \infty$, which contradicts the assumption that $i \notin R$.

For (5), let $v \in V$. Obviously there are only finitely many vertices in U that are adjacent to v via some E_i-edge, where the E_i-degree of v is finite. Since V is finite, (5) follows immediately. ∎

Intuitively, (1)–(3) state the properties matrices A and B should have when considering $A|B$-biregular graphs for the case considered in this subsection, which also allows us to identify a subgraph whose biregularity can be characterized using subsection D.2.1. See Figure 13 for an illustration of the decomposition of matrices A and B. We will use (4) and (5) to identify a corresponding subgraph whose biregularity is characterized using the finite case covered in Theorem 7.6. For an arbitrary graph, we let R and J be as defined in Lemma D.11.

Let C be the matrix obtained by replacing every ∞ entry in B with 0^{+1}. Intuitively, we replace ∞ with some finite value.[11] Let A_3 be the matrix obtained from A

$$A := \left(\begin{array}{c|c} A_1 & 0 \ \mathit{or}\ 0^{+p} \\ \hline A_2 & \underbrace{A_3}_{\text{columns in } J} \end{array}\right) \begin{array}{l} \big\}\ \text{rows not in } R \\ \big\}\ \text{rows in } R \end{array}$$

$$B := \underbrace{\left(\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right)}_{\text{every column contains } \infty \text{ in some row in } R} \begin{array}{l} \big\}\ \text{rows not in } R \\ \big\}\ \text{rows in } R \end{array}$$

FIG. 13. *An illustration of the matrices A and B for the case when there is a complete $A|B$-biregular graph $G = (U, V, E_1, \ldots, E_t)$ with infinitely many vertices on the left-hand side and only finitely many vertices on the right-hand side. Suppose $U = U_1 \uplus \cdots \uplus U_n$ and $V = V_1 \uplus \cdots \uplus V_n$ is the witness partition. R is the set of color i, where $|E_i| = \infty$ and J is the set of column j where U_j is infinite.*

[11] Technically we cannot simply replace ∞ with 0^{+1} since we insist that every periodic entry in a degree matrix has period p. Instead we can replace it with $0^{+p}, 1^{+p}, \ldots, (p-1)^{+p}$ by repeating the columns. For example, a column $\binom{\infty}{a}$ becomes $\binom{0^{+p}}{a}, \binom{1^{+p}}{a}, \ldots, \binom{(p-1)^{+p}}{a}$. We allow the matrix to have 0^{+1} entries.

by keeping only the rows in R and the columns in J. Let B_2 be the matrix obtained from B by keeping only the rows in R and D be the matrix obtained from B_2 by replacing every non-∞ entry with 0.

Let U^* be the set of vertices in $\bigcup_{j\in J} U_j$ adjacent to some $v \in V$ via some E_i-edges, where the E_i-degree of v is finite. For each $j \in J$, let $U_{j,\mathrm{fin}}$ be the set of vertices in U_j with nonzero E_i-degree for some $i \notin R$. Define the sets

$$\begin{aligned} U^{\mathrm{fin}} &:= U^* \cup \bigcup_{j\notin J} U_j \cup \bigcup_{j\in J} U_{j,\mathrm{fin}}, \\ U^{\infty} &:= U - U^{\mathrm{fin}}. \end{aligned}$$

By (4) in Lemma D.11, the set $U_{j,\mathrm{fin}}$ is finite for every $j \in J$. By (5), the set U^* is finite. Thus, the set U^{fin} is finite. By the definition of U^{∞}, a vertex in U^{∞} has nonzero E_i-degree only when $i \in R$. See Figure 14 for an illustration.

The following lemma will provide our reduction.

Lemma D.12. *Suppose $G = ((U, V, E_1, \ldots, E_t)$ is a complete $A|B$-biregular graph with size $\bar{M}|\bar{N}$. Let U^{fin} and U^{∞} be as defined above, G_{fin} denote the induced subgraph $G[U^{\mathit{fin}} \cup V]$, and G_{∞} denote the induced subgraph $G[U^{\infty} \cup V]$. Then*

- *G_{fin} is a (finite) complete $A|C$-biregular graph with size $\bar{K}|\bar{N}$ for some $\bar{K} = (K_1, \ldots, K_m)$, where $K_j = M_j$ if $j \notin J$, and K_j is some finite value if $j \in J$;*
- *G_{∞} is a complete $A_3|D$-biregular graph with size $(\infty, \ldots, \infty)|\bar{N}$.*

Proof. For each vertex $w \in U \cup V$, for each color $i \in [t]$, we say that *the E_i-degree of w is affected in G_{fin} (resp., G_{∞})*, if its E_i-degree in G_{fin} (resp., G_{∞}) is different from its E_i-degree in G. Otherwise, we say that the E_i-degree of w is *unaffected* in G_{fin} (resp., G_{∞}).

Towards proving the first bullet item, note that for each $i \in [t]$, the E_i-degree of every vertex u is unaffected in G_{fin}, since V is still the set of vertices on the left-hand side of G_{fin}. On the other hand, for each vertex $v \in V$, and color $i \in [t]$, if the E_i-degree of v is finite in G, then its E_i-degree is unaffected in G_{fin}. This is because if $(u, v) \in E_i$ and the E_i-degree of v is finite, then by definition, $u \in U^*$ and, hence, $u \in U^{\mathrm{fin}}$. So, the E_i-degree of v is affected in G_{fin} only when the E_i-degree of v is ∞

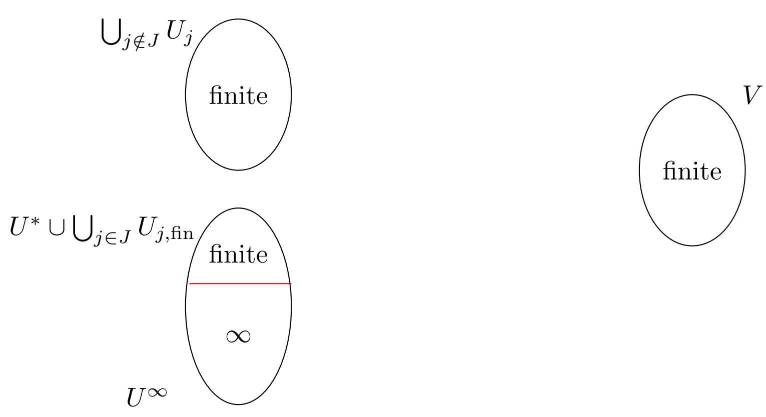

Fig. 14. *Illustration of the set $\bigcup_{j\notin J} U_j$, $U^* \cup \bigcup_{j\in J} U_{j,\mathit{fin}}$, and U^{∞} with their sizes. The set U^{fin} is the union $U^* \cup \bigcup_{j\in J} U_{j,\mathit{fin}} \cup \bigcup_{j\notin J} U_j$, which is finite. In the induced graph $G_{\mathit{fin}} = G[U^{\mathit{fin}} \cup V]$, the vertices in V have finite total degrees. In the induced graph $G_{\infty} = G[U^{\infty} \cup V]$, there are no E_i-edges for $i \notin R$. Note: color appears only in the online article.*

in G, which has now become finite in G_{fin}. Since every ∞ entry in B has now becomes 0^{+1} in C, it follows immediately that G_{fin} is a complete $A|C$-biregular graph.

Turning to the second bullet item, the E_i-degree of every vertex in U^{∞} is obviously unaffected in G_{∞}. Moreover, every vertex in U^{∞} has nonzero E_i-degree in G only when $i \in R$. Thus, the colors of the edges in G_{∞} are only those in R. For every vertex $v \in V$,

- if its E_i-degree is ∞ in G, its E_i-degree is unaffected in G_{∞};
- if its E_i-degree is finite in G, its E_i-degree becomes 0 in G_{∞}.
  This is because if u and v are adjacent via an E_i-edge and the E_i-degree of v is finite, then $u \in U^*$, hence, $u \in U^{\text{fin}}$.

Since every finite entry in B_2 becomes 0 in D, it follows immediately that G_{∞} is $A_3|D$-biregular. ∎

Lemma D.12 reduces characterization of the sizes of $A|B$-biregular graphs to characterizations of finite complete biregular graphs (which we have provided in the body) and characterization of infinite $A_3|D$-biregular graphs, whose sizes are of the form $(\infty,\ldots,\infty)|\bar{N}$, i.e., the components in the size vectors on the left are all ∞ and every entry in D is either 0 or ∞.

We will next define formulas that capture the sizes $\bar{M}|\bar{N}$ of complete $A|B$-biregular graphs, assuming that the left-hand side has infinitely many vertices and the right-hand side has finitely many vertices. Let R be the set of colors i where the number of E_i-edges is infinite. and $J \subseteq [m]$ be the set of indexes j, where $M_j = \infty$.

Let $\bar{x} = (x_1,\ldots,x_m)$, $\bar{y} = (y_1,\ldots,y_n)$, and $\bar{z} = (z_1,\ldots,z_m)$. Let $\xi^{J,R}_{A|B}(\bar{x},\bar{y})$ be the formula

$$\|\bar{x}^T\| = \infty \ \wedge\ \|\bar{y}^T\| \neq \infty \tag{D.6}$$

$$\wedge \quad \bigwedge_{i\in R} \|\bar{x}^T\|_{\text{nz}(A_{i,*})} = \infty \ \wedge \ \bigwedge_{i\notin R} \|\bar{x}^T\|_{\text{nz}(A_{i,*})} \neq \infty \tag{D.7}$$

$$\wedge \quad \bigwedge_{j\in J} x_j = \infty \ \wedge \ \bigwedge_{j\notin J} x_j \neq \infty \ \wedge \tag{D.8}$$

$$\wedge \quad \exists \bar{z}\ \ \mathsf{c\text{-}bireg}_{A|C}(\bar{z},\bar{y}) \ \wedge \ \|\bar{z}^T\| \neq \infty \ \wedge \ \bigwedge_{j\notin J} z_i = x_i \tag{D.9}$$

$$\wedge \quad \xi_{A_3|D}(\bar{y}), \tag{D.10}$$

where formula $\mathsf{c\text{-}bireg}_{A|C}(\bar{z},\bar{y})$ captures the sizes of the finite complete $A|C$-biregular graph as defined in Theorem 7.6 and $\xi_{A_3|D}(\bar{y})$ is as defined in (D.5).

Intuitively, (D.6)–(D.8) state that there are infinitely many vertices on the left and only finitely many on the right, and that R and J are as defined above. The next lemma follows immediately from Lemma D.8, Theorem 7.6, and Lemma D.12.

Lemma D.13. *For every pair of matrices A, B and every pair of size vectors $\bar{M}$, $\bar{N}$ with infinitely many vertices on the left, finitely many on the right, R and J defined as above, the formula $\xi^{J,R}_{A|B}(\bar{M},\bar{N})$ holds in $\mathcal{N}_{\infty}$ if and only if there is a complete $A|B$-biregular graph of size $\bar{M}|\bar{N}$.*

To wrap up this subsection, we define the formula:

$$\mathsf{c\text{-}bireg}^{\infty,\text{fin}}_{A|B}(\bar{x},\bar{y}) \ := \bigvee_{J\subseteq[m],R\subseteq[t]} \xi^{J,R}_{A|B}(\bar{x},\bar{y})$$

that captures the sizes of all possible $A|B$-biregular graphs where the left-hand side has infinitely many vertices and the right-hand side has finitely many vertices.

Remark D.14. By Lemma 8.1 for finite graphs, $\mathsf{c\text{-}bireg}_{A|C}(\bar{z},\bar{y})$ is a disjunction of conjunctions of $O(mnt^4\delta(A,B)^4)$ (in)equations. By definition, $\xi_{A_3|D}(\bar{y})$ is a disjunction of conjunctions of $O(tmn)$ (in)equations. Thus, $\mathsf{c\text{-}bireg}^{\infty,\mathrm{fin}}_{A|B}(\bar{x},\bar{y})$ is a disjunction of conjunctions of $O(mnt^4\delta(A,B)^4)$ (in)equations.

For arbitrary degree matrices A and B, we can define a formula $\mathsf{c\text{-}bireg}_{A|B}(\bar{x},\bar{y})$ capturing the sizes of all possible $A|B$-biregular graphs as a disjunction of the formulas for each of the following four cases:

- Both sides have finitely many vertices, which by Lemma 8.1 is a disjunction of conjunctions of $O(mnt^4\delta(A,B)^4)$ (in)equations.
- The left-hand side has infinitely many vertices and the right-hand side has finitely many vertices, which as explained above is a disjunction of conjunctions of $O(mnt^4\delta(A,B)^4)$ (in)equations.
- The left-hand side has finitely many vertices and the right-hand side has infinitely many vertices, which is symmetric to the previous case.
- Both sides have infinitely many vertices.
  By Remark D.5, the formula when the degree matrices are simple matrices is a disjunction of conjunctions of $O(2^tt^4\delta(A,B))$ (in)equations. Since the transformation from nonsimple to simple requires a blowup of the $O(mn)$ factor, this case is a disjunction of conjunctions of $O(mn2^tt^4\delta(A,B)^4)$ (in)equations.

We conclude that $\mathsf{c\text{-}bireg}_{A|B}(\bar{x},\bar{y})$ can be expressed as a disjunction of conjunctions of $O(mn2^tt^4\delta(A,B)^4)$ (in)equations.

Appendix E. The extension of section 7 to the general case. In this appendix we explain briefly how to extend the reduction from nonsimple degree matrices to simple degree matrices in section 7, now allowing the degree matrices to contain ∞ entries and the sizes of the partitions to be infinite. This reduction is only applied to the finite case and case 1 from the prior appendix, where there are infinite degree vertices on both sides. In the case where exactly one side had an infinite degree vertex, we did not make use of the simple restriction. The reduction we give below can actually apply to all cases, but making use of it in the last case above would not give the desired complexity.

We need to modify the definition of behavior functions in Definition 7.4 a little bit, to take into account that the entry in a degree matrix can be ∞.

DEFINITION E.1. *For each $j \in [m]$, we define* a behavior function of column j in A *to be a function $g : [t] \times [n] \to \{0,1,\ldots,q,0^{+p},1^{+p},\ldots,q^{+p},\infty\}$ such that the following hold:*

- $A_{*,j} = \begin{pmatrix} g(1,1)+\cdots+g(1,n) \\ g(2,1)+\cdots+g(2,n) \\ \vdots \\ g(t,1)+\cdots+g(t,n) \end{pmatrix}$;
- *for each color $i \in [t]$, if $A_{i,j}$ is a fixed entry, then $g(i,1),\ldots,g(i,n)$ are all fixed entries;*
- *for each color $i \in [t]$, if $A_{i,j}$ is a periodic entry, then $g(i,1),\ldots,g(i,n)$ are all periodic entries;*
- *for each color $i \in [t]$, if $A_{i,j}$ is an ∞ entry, then $g(i,1),\ldots,g(i,n)$ are all periodic entries and at least one of them is ∞.*

Note that the difference between Definition 7.4 and Definition E.1 is the addition of the third item, where the entry $A_{i,j}$ can be ∞. The definition of a behavior function of column j' in B is also modified in a similar manner. The reduction from nonsimple matrices to simple matrices can now be obtained in exactly the same manner as in subsection 7.2.

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